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ANNEX

Simpson
OEF

Rule for extracting the square root acco
ding to Webber.

Rule. 1 Range the quantities according to the di
sions of some letter and set the root of the firs
term in the plase of a quotient for the first t
of the root required.

2. subtract the square of this root from th
first term and bring down the two next ter
to the remainder for a dividend.

3. Divide the first term of the dividend by d
the root and set the quotient for the next
term of the root.

4. Multiply the divisor and the last term of t
root by that last term and subtract the
product from the dividend: and so on as
Arithmetic.

Phil. 1809 [illegible]

A

TREATISE

OF

ALGEBRA:

WHEREIN THE

PRINCIPLES ARE DEMONSTRATED,

AND APPLIED

IN MANY USEFUL AND INTERESTING INQUIRIES, AND IN THE RESOLUTION OF A GREAT VARIETY OF PROBLEMS OF DIFFERENT KINDS.

TO WHICH IS ADDED,

THE GEOMETRICAL CONSTRUCTION OF A GREAT NUMBER OF LINEAR AND PLANE PROBLEMS,

WITH THE METHOD OF RESOLVING THE SAME NUMERICALLY.

BY THOMAS SIMPSON, F. R. S.

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THE

AUTHOR'S PREFACE

TO THE

SECOND EDITION.

THE motives that first gave birth to the ensuing work, were not so much any extravagant hopes the author could form to himself of greatly extending the subject by the addition of a large variety of new improvements (though the reader will find many things here that are nowhere else to be met with), as an earnest desire to see a subject of such general importance established on a clear and rational foundation, and treated as a science, capable of demonstration, and not a myterious art, as some authors, themselves, have thought proper to term it.

How well the design has been executed, must be left for others to determine. It is possible that the pains here taken, to reduce the fundamental principles, as well as the more difficult parts of the subject to a demonstration, may be looked upon, by some, as rather tending to throw new difficulties in the way of a learner, than to the facilitating of his progress. In order to gratify, as far as might be, the inclination of this class of readers, the demonstrations are now given by themselves, in the manner of notes (so as to be taken or omitted at pleasure); though the author cannot by any means be induced to think that time lost to a learner which is taken up in comprehending the grounds whereon he is to raise his superstructure: his progress may, indeed, at first, be a little retarded; but the *real* knowledge he thence acquires will abundantly compensate his trouble, and enable him to proceed, afterwards, with certainty and success, in matters of greater difficulty, where authors, and their rules, can yield him no assistance, and he has nothing to depend upon but his own observation and judgment.

This second edition has many advantages over the former, as well with respect to a number of new subjects and improvements, interspersed throughout the whole, as in the order and disposition of the elementary parts: in which particular regard has been had to the capacities of young beginners. The work, as it now stands, will, the author flatters himself, be found equally plain and comprehensive, so as to answer, alike, the purpose of the lower and of the more experienced class of readers.

P. S. *The great reputation of* Mr. SIMPSON'S TREATISE *of* ALGEBRA, *and the favourable reception it has universally met with since its first publication, and which testifies it to be the best elementary work upon the subject, has induced the proprietor to have this edition carefully revised and corrected throughout by a very eminent mathematician: he therefore trusts it will be found as worthy the approbation of the public as any former edition, or as if revised by the author himself.*

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SECTION XVIII.

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A

TREATISE

OF

ALGEBRA.

SECTION I.

Of Notation.

ALGEBRA is that science which teaches, in a general manner, the relation and comparison of abstract quantities: by means whereof such questions are resolved whose solutions would be sought in vain from common arithmetic.

In algebra, otherwise called *specious arithmetic*, numbers are not expressed as in the common notation, but every quantity, whether given or required, is commonly represented by some letter of the alphabet; the given ones, for distinction sake, being usually denoted by the initial letters a, b, c, d, &c.; and the unknown or required ones by the final letters u, w, x, y, &c. There are, moreover, in algebra, certain signs or notes made use of to show the relation and dependence of quantities one upon another, whose signification the learner ought, first of all, to be made acquainted with.

The sign + signifies that the quantity to which it is prefixed is to be added. Thus $a+b$ shows that the number represented by b is to be added to that represented by a, and expresses the sum of those numbers; so that if a was 5, and b 3, then would $a+b$ be $5+3$, or 8. In like manner, $a+b+c$ denotes the number arising by adding all the three numbers a, b, and c together.

Note. A quantity which has no prefixed sign (as the leading quantity a in the above examples) is always understood to have the sign + before it: so that a signifies the same as $+a$; and $a+b$ the same as $+a+b$.

The sign — signifies that the quantity which it precedes is to be subtracted. Thus $a-b$ shows that the quantity represented by b is to be subtracted from that represented by a, and expresses the difference of a and b: so that if a was 5, and b 3, then would $a-b$ be $5-3$, or 2. In like manner, $a+b-c-d$ represents the quantity which arises by taking the numbers c and d from the sum of the other two numbers a and b: as, if a was 7, b 6, c 5, and d 3, then would $a+b-c-d$ be $7+6-5-3$, or 5.

The notes + and — are usually expressed by the words *plus* (or *more*), and *minus* (or *less*). Thus, we read $a+b$, *a plus b*; and $a-b$, *a minus b*.

Moreover, those quantities to which the sign + is prefixed are called *positive* (or *affirmative*); and those to which the sign — is prefixed, *negative*.

The sign × signifies that the quantities between which it stands are to be multiplied together. Thus $a\times b$ denotes that the quantity a is to be multiplied by the quantity b, and expresses the product of the quantities so multiplied; and $a\times b\times c$ expresses the product arising by multiplying the quantities a, b, and c, continually together: thus, likewise, $\overline{a+b}\times c$ denotes the product of the compound quantity $a+b$ by the simple quantity c; and $\overline{a+b+c}\times\overline{a-b+c}\times\overline{a+c}$ represents the product which arises by multiplying the three compound quantities $a+b+c$, $a-b+c$, and $a+c$ continually together; so that if a was 5, b 4, and c 3, then would $\overline{a+b+c} \times \overline{a-b+c} \times \overline{a+c}$ be $12 \times 4 \times 8$, which is 384.

But when quantities denoted by single letters are to be multiplied together, the sign × is generally omitted, or only understood; and so ab is made to signify the same as $a \times b$; and abc the same as $a \times b \times c$.

It is likewise to be observed, that when a quantity is to be multiplied by itself, or raised to any power, the usual method of notation is to draw a line over the given quantity, and at the end thereof place the exponent of the power. Thus $\overline{a+b}\rvert^{2}$ denotes the same as $\overline{a+b} \times \overline{a+b}$, *viz.* the second power (or square) of $a+b$ considered as one quantity: thus, also, $\overline{ab+bc}\rceil^{3}$ denotes the same as $\overline{ab+bc} \times \overline{ab+bc} \times \overline{ab+bc}$, *viz.* the third power (or cube) of the quantity $ab+bc$.

But, in expressing the powers of quantities represented by single letters, the line over the top is commonly omitted; and so a^{2} comes to signify the same as aa or $a \times a$, and b^{3} the same as bbb or $b \times b \times b$: whence also it appears that $a^{2}b^{3}$ will signify the same as $aabbb$; and $a^{5}c^{2}$ the same as $aaaaacc$; and so of others.

The note . (or a full point), and the word *into*, are likewise used instead of ×, or as marks of multiplication.

Thus $\overline{a+b} \, . \, \overline{a+c}$ and $\overline{a+b}$ *into* $\overline{a+c}$, both signify the same thing as $\overline{a+b} \times \overline{a+c}$, namely, the product of $a+b$ by $a+c$.

The sign ÷ is used to signify that the quantity preceding it is to be divided by the quantity which comes after it: thus $c \div b$ signifies that c is to be divided by b; and $\overline{a+b} \div \overline{a-c}$, that $a+b$ is to be divided by $a-c$.

Also the mark) is sometimes used as a note of division; thus, $a+b)\ ab$, denotes that the quantity ab is to be divided by the quantity $a+b$; and so of others. But the division of algebraic quantities is most commonly expressed by writing down the divisor under the dividend with a line between them, in the manner of a vulgar fraction. Thus $\frac{c}{b}$ represents the quantity arising by dividing c by b; and $\frac{a+b}{a-c}$ denotes the quantity arising by dividing $a+b$ by $a-c$. Quantities thus expressed are called algebraic fractions; whereof the upper part is called the numerator, and the lower the denominator, as in vulgar fractions.

The sign $\sqrt{}$ is used to express the square root of any quantity to which it is prefixed: thus $\sqrt{25}$ signifies the square root of 25 (which is 5, because 5×5 is 25): thus also $\sqrt{ab}$ denotes the square root of ab; and $\sqrt{\frac{ab+bc}{d}}$ denotes the square root of $\frac{ab+bc}{d}$, or of the quantity which arises by dividing $ab+bc$ by d; but $\frac{\sqrt{ab+bc}}{d}$ (because the line which separates the numerator from the denominator is drawn below $\sqrt{}$) signifies that the square root of $ab+bc$ is to be *first* taken, and *afterwards* divided by d: so that if a was 2, b 6, c 4, and d 9, then would $\frac{\sqrt{ab+bc}}{d}$ be $\frac{\sqrt{36}}{9}$ or $\frac{6}{9}$; but $\sqrt{\frac{ab+bc}{d}}$ is $\sqrt{\frac{36}{9}}$, or $\sqrt{4}$, which is 2.

The same mark $\sqrt{}$, with a figure over it, is also used to express the cube, or biquadratic root, &c. of any quantity: thus $\sqrt[3]{64}$ represents the cube root of 64 (which is 4, because $4 \times 4 \times 4$ is 64), and $\sqrt[3]{ab+cd}$ the cube root of $ab+cd$; also $\sqrt[4]{16}$ denotes the biquadratic root of 16 (which is 2, because $2 \times 2 \times 2 \times 2$ is 16); and $\sqrt[4]{ab+cd}$ denotes the biquadratic root of $ab+cd$; and so of others. Quantities thus expressed are called radical quantities, or surds; whereof those consisting of one term only, as $\sqrt{a}$ and $\sqrt{ab}$, are called *simple surds;* and those consisting of several terms, or members, as $\sqrt{a^2-b^2}$ and $\sqrt[3]{a^2-b^6+bc}$, *compound surds.*

Besides this way of expressing radical quantities, (which is chiefly followed) there are other methods made use of by different authors; but the most commodious of all, and best suited to practice, is that, where the root is designed by a vulgar fraction, placed at the end of a line drawn over the quantity given. According to this notation, the square root is designed by the fraction $\frac{1}{2}$, the cube root by $\frac{1}{3}$, and the biquadratic root by $\frac{1}{4}$, &c. Thus $\overline{a}\rceil^{\frac{1}{2}}$ expresses the same thing with $\sqrt{a}$, *viz.* the square root of a; and $\overline{a^2+ab}\rceil^{\frac{1}{3}}$ the same as $\sqrt[3]{a^2+ab}$, that is, the cube root of a^2+ab: also $\overline{a}\rceil^{\frac{2}{3}}$ denotes the square of the cube root of a; and $\overline{a+z}\rceil^{\frac{7}{4}}$ the seventh power of the biquadratic root of $a+z$; and so of others. But it is to be observed, that, when the root of a quantity represented by a single letter is to be expressed, the line over it may be neglected; and so $a^{\frac{1}{2}}$ will signify the same as $\overline{a}\rceil^{\frac{1}{2}}$, and $b^{\frac{1}{3}}$ the same as $\overline{b}\rceil^{\frac{1}{3}}$ or $\sqrt[3]{b}$. The number, or fraction, by which the power, or root of any quantity is thus designed is called its index, or exponent.

The mark $=$ *(called the sign of equality) is used to signify that the quantities standing on each side of it are equal.* Thus $2+3=5$, shows that 2 more 3 is equal to 5; and $x=a-b$, shows that x is equal to the difference of a and b.

The note $::$ *signifies that the quantities between which it stands are proportional:* as, $a:b::c:d$, denotes that a is in the same proportion to b, as c is to d; or that if a be twice, thrice, or four times, &c. as great as b, then accordingly c is twice, thrice, or four times, &c. as great as d.

To what has been thus far laid down on the signification of the signs and characters used in the algebraic notation, we may add what follows, which is equally necessary to be understood.

When any quantity is to be taken more than once, the number is to be prefixed, which shows how many times it is to be taken: thus $5a$ denotes that the quantity a is to be taken five times; and $3bc$ stands for three times bc, or the quantity which arises by multiplying bc by 3: also $7\sqrt{a^2+b^2}$ signifies that $\sqrt{a^2+b^2}$ is to be taken seven times; and so of others.

The numbers thus prefixed are called coefficients; and that quantity which stands without a coefficient is always understood to have a unit prefixed, or to be taken once, and no more.

Those quantities are said to be *like* that are expressed by the same letters under the same powers, or which differ only in their coefficients: thus $3bc$, $5bc$, and $8bc$ are *like* quantities; and the same is to be understood of the radicals $2\sqrt{\frac{b+c}{a}}$ and $7\sqrt{\frac{b+c}{a}}$. But *unlike* quantities are those which are expressed by different letters, or by the same letters under different powers: thus $2ab$, $2abc$, $5ab^2$, and $3ba^2$ are all *unlike*.

When a quantity is expressed by a single letter, or by several single letters joined together in multiplication without any sign between them, as a, or $2ab$, it is called a *simple* quantity.

But that quantity which consists of two or more such simple quantities, connected by the signs + or —, is called a *compound* quantity: thus $a-2ab+5abc$ is a *compound* quantity; whereof the *simple* quantities a, $2ab$, and $5abc$ are called the terms or members.

The letters by which any simple quantity is expressed may be ranged according to any order at pleasure, and yet the signification continue the same: thus ab may be written ba; for ab denotes the product of a by b, and ba the product of b by a; but it is well known, that when two numbers are to be multiplied together, it matters not which of them is made the multiplicand, nor which the multiplier, the product, either way, coming out the same. In like manner it will appear that *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*, all express the same thing, and may be used indifferently for each other, as will be demonstrated further on; but it will be sometimes found convenient, in long operations, to place the several letters according to the order which they have in the alphabet.

Likewise the several members or terms of which any quantity is composed may be disposed according to any order at pleasure, and yet the signification be nowise affected thereby. Thus $a - 2ab + 5a^2b$ may be written $a + 5a^2b - 2ab$, or $-2ab + a + 5a^2b$, &c.; for all these represent the same thing, *viz.* the quantity which remains, when, from the sum of a and $5a^2b$, the quantity $2ab$ is deducted.

Here follow some examples wherein the several forms of notation hitherto explained are promiscuously concerned, and where the signification of each is expressed in numbers.

Suppose $a = 6$, $b = 5$, and $c = 4$; then will

$$a^2 + 3ab - c^2 = 36 + 90 - 16 = 110,$$

$$2a^3 - 3a^2b + c^3 = 432 - 540 + 64 = -44,$$

$$a^2 \times \overline{a + b} - 2abc = 36 \times 11 - 240 = 156,$$

$$\frac{a^3}{a + 3c} + c^2 = \frac{216}{18} + 16 = 12 + 16 = 28,$$

$$\sqrt{2ac + c^2}\,(\text{or } \overline{2ac + c^2}\,\rceil^{\frac{1}{2}}) = \sqrt{64} = 8 \text{ (for } 8 \times 8 = 64),$$

$$c^{\frac{1}{2}} + \frac{2bc}{\sqrt{2ac + c^2}} = 2 + \frac{40}{8} = 7,$$

$$\frac{a^2 - \sqrt{b^2 - ac}}{2a - \sqrt{b^2 + ac}} = \frac{36 - 1}{12 - 7} = \frac{35}{5} = 7,$$

$$\sqrt{b^2 - ac} + \sqrt{2ac + c^2} = 1 + 8 = 9,$$

$$\sqrt{b^2 - ac + \sqrt{2ac + c^2}} = \sqrt{25 - 24 + 8} = 3.$$

This method of explaining the signification of quantities I have found to be of good use to young beginners; and would recommend it to such as are desirous of making proficiency in the subject, to get a clear idea of what has been thus far delivered, before they proceed farther.

SECTION II.

Of Addition.

ADDITION, in algebra, is performed by connecting the quantities by their proper signs, and joining into one sum such as can be united: for the more ready effecting of which, observe the following rules.

1°. *If, in the quantities to be added, there be terms that are like, and have all the same sign, add the coefficients of those terms together, and to their sum adjoin the letters common to each term, prefixing the common sign.*

Thus	$5a$	And	$5a+7b$	Also	$5a-7b$
added to	$3a$	added to	$7a+3b$	added to	$7a-3b$
makes	$8a.$	makes	$12a+10b.$	makes	$12a-10b.$

Hence likewise the sum of

$$\left\{\begin{array}{l} 2\sqrt{ab}+7\sqrt{bc} \\ 3\sqrt{ab}+2\sqrt{bc} \\ 6\sqrt{ab}+9\sqrt{bc} \end{array}\right.$$

will be $11\sqrt{ab}+18\sqrt{bc}$

And the sum of

$$\left\{\begin{array}{l} \frac{2b}{a}-\frac{3d}{c} \\ \frac{5b}{a}-\frac{7d}{c} \end{array}\right.$$

will be $\frac{7b}{a}-\frac{10d}{c}$

The reasons on which the preceding operations are grounded will readily appear by reflecting a little on the nature and signification of the quantities to be added: for, with regard to the first example, where $3a$ is to be added to $5a$, it is plain that three times any quantity whatever, added to five times the same quantity, must make eight times that quantity: therefore $3a$, or three times the quantity denoted by a, being added to $5a$, or five times the same quantity, the sum must consequently

2°. *When, in the quantities to be added, there are like terms, whereof some are affirmative and others negative, add together the affirmative terms (if there be more than one), and do the same by the negative ones; then take the difference of the two sums (not regarding the signs) by subtracting the coefficient of the lesser from that of the greater, and adjoin the letters common to each; to which difference prefix the sign of the greater.*

Examples of this rule may be as follows.

$$\begin{array}{lr}
1. & 12a - 5b \\
 & -3a + 2b \\
\hline
\text{Sum} & 9a - 3b
\end{array}
\qquad
\begin{array}{lr}
2. & -3ab + 5bc \\
 & +7ab - 9ac \\
\hline
\text{Sum} & 4ab - 4bc
\end{array}$$

$$\begin{array}{lr}
3. & 6ab + 12bc - 8cd \\
 & -7ab - 9bc + 3cd \\
 & -2ab - 5bc + 12cd \\
\hline
\text{Sum} & -3ab - 2bc + 7cd
\end{array}
\qquad
\begin{array}{lr}
4. & 5\sqrt{ab} - 7\sqrt{bc} + 8d \\
 & 3\sqrt{ab} + 8\sqrt{bc} - 12d \\
 & 7\sqrt{ab} + 3\sqrt{bc} + 9d \\
\hline
\text{Sum} & 15\sqrt{ab} + 4\sqrt{bc} + 5d
\end{array}$$

$$\begin{array}{lr}
5. & 12abc - 16abd + 25acd - 72bcd \\
 & 16abc + 12abd + 20acd - 18bcd \\
 & -13abc - 26abd - 15acd + 12bcd \\
 & 82abc + 18abd - 10acd - 16bcd \\
\hline
\text{Sum} & 47abc - 12abd + 20acd - 94bcd
\end{array}$$

---

make $8a$, or eight times that quantity. From whence, as the sum of any two quantities is equal to the sum of all their parts, the reason of the second case, or example, is likewise obvious. But as to the third, where the given quantities are $5a - 7b$ and $7a - 3b$, we are to consider, that, if the two quantities to be added together had been exactly $5a$ and $7a$ (which are the two leading terms), the sum would then have been just $12a$; but, since the former quantity wants $7b$ of $5a$, and the latter $3b$ of $7a$, their sum must, it is evident, want both $7b$ and $3b$ of $12a$; and therefore be equal to $12a - 10b$, that is, equal to what remains, when the sum of the defects is deducted. And, by the very same way of arguing, it is easy to conceive that the sum which arises by adding any number

6.
$$\begin{array}{rl} & \frac{5a}{b} - \frac{3cc}{a} + 7\sqrt{\frac{bc}{a}} - 9\sqrt{\frac{ab+cc}{a}} \\ & \frac{8a}{b} + \frac{7cc}{a} - 12\sqrt{\frac{bc}{a}} + 6\sqrt{\frac{ab+cc}{a}} \\ \hline \text{Sum} & \frac{13a}{b} + \frac{4cc}{a} - 5\sqrt{\frac{bc}{a}} - 3\sqrt{\frac{ub+cc}{a}} \end{array}$$

In the last example, and all others where fractional and radical quantities are concerned, every such quantity, exclusive of its coefficient, is to be treated in all respects like a simple quantity expressed by a single letter.

3°. *When, in the quantities to be added, there are terms without others like to them, write them down with their proper signs.*

$$\begin{array}{ll} \text{Thus} & a+2b \\ \text{added to} & \underline{3c+d} \\ \text{makes} & a+2b+3c+d \end{array} \qquad \begin{array}{ll} \text{And} & aa+bb \\ \text{added to} & \underline{a+b} \\ \text{makes} & aa+bb+a+b \end{array}$$

Here follow a few examples for the learner's exercise, wherein all the three foregoing rules take place promiscuously.

1.
$$\begin{array}{rl} & 2aa + 3ab + 8cc + d^2 \\ & 5aa - 7ab + 5cc - d^3 \\ & -2aa + 4ab + 3cc + 30 \\ \hline \text{Sum} & 5aa \; * \; + 16cc + d^2 - d^3 + 30. \end{array}$$

---

of quantities together, will be equal to the sum of all the affirmative terms diminished by the sum of all the negative ones (considered independent of their signs): from whence the reason of the second general rule is apparent. As to the case where the quantities are unlike, it is plain that such quantities cannot be united into one, or otherwise added than by their signs: thus, for example, let a be supposed to represent a crown, and b a shilling; then the sum of a and b can be neither $2a$ nor $2b$, that is, neither two crowns nor two shillings, but one crown *plus* one shilling, or $a+b$.

$$\begin{array}{lllll}
2. & 5\sqrt{ax} & -\ 8\sqrt{aa-xx} & +\ 12\sqrt{aa+4xx} \\
 & 8\sqrt{ax} & +\ 15\sqrt{aa-xx} & -\ 8\sqrt{aa+4xx} \\
 & 6\sqrt{ax} & -\ 7\sqrt{aa-xx} & +\ 10\sqrt{aa+4xx} \\
\hline
\text{Sum} & 19\sqrt{ax} & * & +\ 14\sqrt{aa+4xx}
\end{array}$$

$$\begin{array}{ll}
3. & 2a^2 - 3ab + 2b^3 - 3a^2 \\
 & 3b^3 - 2a^2 + a^3 - 5c^3 \\
 & 4c^3 - 2b^3 + 5ab + 100 \\
 & 20ab + 16a^2 - bc - 80 \\
\hline
\text{Sum} & 13a^2 + 22ab + 3b^3 + a^3 - c^3 + 20 - bc.
\end{array}$$

SECTION III.

Of Subtraction.

SUBTRACTION, in algebra, is performed by changing all the signs of the subtrahend (or conceiving them to be changed), and then connecting the quantities, as in addition.

$$\begin{array}{llll}
\text{Ex. 1. From} & 8a+5b & \text{Ex. 2. From} & 8a+5b \\
\text{take} & 5a+3b & \text{take} & 5a-3b \\
\hline
\text{Rem.} & 3a+2b. & \text{Rem.} & 3a+8b. \\
\text{Ex. 3. From} & 8a-5b & \text{Ex. 4. From} & 8a-5b \\
\text{take} & 5a+3b & \text{take} & 5a-3b \\
\hline
\text{Rem.} & 3a-8b. & \text{Rem.} & 3a-2b.
\end{array}$$

In the second example, conceiving the signs of the subtrahend to be changed to their contrary, that of $3b$ becomes $+$; and so the signs of $3b$ and $5b$ being alike, the coefficients 3 and 5 are to be added together, by case 1 of addition. The same thing happens in the third example; since the sign of $3b$, when changed, is $-$, and therefore the same with that of $5b$. But, in the fourth example, the signs of $3b$ and $5b$, after that of $3b$ is changed, being unlike, the difference of the coefficients must be taken, conformable to case 2 in addition.

Other examples in subtraction may be as follows:

$$\begin{array}{ll} \text{From} & 20ax + 5bc - 7aa \\ \text{take} & 12ax - 3bc - 5aa \\ \hline \text{Rem.} & 8ax + 8bc - 2aa. \end{array} \qquad \begin{array}{ll} \text{From} & 7\sqrt{ax} + 9\sqrt{by} \\ \text{take} & -5\sqrt{ax} + 12\sqrt{by} \\ \hline \text{Rem.} & 12\sqrt{ax} - 3\sqrt{by}. \end{array}$$

$$\begin{array}{ll} \text{From} & 6\sqrt{aa - xx} + 10\sqrt[3]{a^3 - x^3} - 7\sqrt{\frac{aa}{c}} \\ \text{take} & 9\sqrt{aa - xx} + 15\sqrt[3]{a^3 - x^3} - 9\sqrt{\frac{aa}{c}} \\ \hline \text{Rem.} & -3\sqrt{aa - xx} + 25\sqrt[3]{a^3} - x^3 + 2\sqrt{\frac{aa}{c}} \end{array}$$

$$\begin{array}{ll} \text{From} & 7a^2 - \frac{5a}{c} + 6\sqrt{\frac{ax}{c}} + d \\ \text{take} & a^2 + \frac{8a}{c} - \sqrt{\frac{ax}{c}} + b \\ \hline \text{Rem.} & 6a^2 - \frac{13a}{c} + 7\sqrt{\frac{ax}{c}} + d - b. \end{array}$$

In this last example, the quantity a^2 in the subtrahend being without a coefficient, a unit is to be understood; for $1a^2$ and a^2 mean the same thing. The like is to be observed in all other similar cases.

The grounds of the general rule for the subtraction of algebraic quantities may be explained thus: let it be here required to subtract $5a - 3b$ from $8a + 5b$ (as in ex. 2). It is plain, in the first place, that if the affirmative part $5a$ were alone to be subtracted, the remainder would then be $8a + 5b - 5a$; but, as the quantity actually proposed to be subtracted is less than $5a$, by $3b$, too much has been taken away by $3b$; and therefore the true remainder will be greater than $8a + 5b - 5a$, by $3b$; and so will be truly expressed by $8a + 5b - 5a + 3b$: wherein the signs of the two last terms are both contrary to what they were given in the subtrahend; and where the whole, by uniting the like terms, is reduced to $3a + 8b$, as in the example.

SECTION IV.

Of Multiplication.

BEFORE I proceed to lay down the necessary rules for multiplying quantities one by another, it may be proper to premise the following particulars, in order to give the learner a clear idea of the reason and certainty of such rules.

First, then, it is to be observed, that when several quantities are to be multiplied continually together, the result, or product, will come out exactly the same, multiply them according to what order you will. Thus $a \times b \times c$, $a \times c \times b$, $b \times c \times a$, &c. have all the same value, and may be used indifferently: to illustrate which we may suppose $a = 2$, $b = 3$, and $c = 4$; then will $a \times b \times c = 2 \times 3 \times 4 = 24$; $a \times c \times b = 2 \times 4 \times 3 = 24$; and $b \times c \times a = 3 \times 4 \times 2 = 24$.

Secondly. *If any number of quantities be multiplied continually together, and any other number of quantities be also multiplied continually together, and then the two products one into the other, the quantity thence arising will be equal to the quantity that arises by multiplying all the proposed quantities continually together.* Thus will $abc \times de = a \times b \times c \times d \times e$; so that, if a were $= 2$, $b = 3$, $c = 4$, $d = 5$, $e = 6$, then would $abc \times de = 24 \times 30 = 720$, and $a \times b \times c \times d \times e = 2 \times 3 \times 4 \times 5 \times 6 = 720$. The general demonstration of these observations is given below in the notes.

The following demonstrations depend on this principle, *that if two quantities, whereof the one is n times as great as the other (n being any number at pleasure), be multiplied by one and the same quantity, the product, in the one case, will* also *be n times as great as in the other.* The greater quantity may be conceived to be divided into n parts, equal, each, to the lesser quantity; and the product of each part (by the given multiplier) will

The multiplication of algebraic quantities may be considered in the seven following cases.

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be equal to that of the said lesser quantity; therefore the sum of the products of all the parts which make up the whole greater product, must necessarily be n times as great as the lesser product, or the product of one single part, alone.

This being premised, it will readily appear, in the first place, that $b \times a$ and $a \times b$ are equal to each other: for, $b \times a$ being b times as great as $1 \times a$ (*because the multiplicand is b times as great*) it must therefore be equal to $1 \times a$ (or a) repeated b times, that is, equal to $a \times b$, *by the definition of multiplication.*

In the same manner, the equality of all the variations, or products, abc, bac, acb, cab, bca, cba (where the number of factors is 3) may be inferred: for those that have the last factors the same (*which I call of the same class*) are manifestly equal, being produced of equal quantities multiplied by the same quantity: and, to be satisfied that those of *different classes*, as abc and acb, are likewise equal, we need only consider, that, since $ac \times b$ is c times as great as $a \times b$ (because the multiplicand is c times as great) it must therefore be equal to $a \times b$ taken c times, that is, equal to $a \times b \times c$, *by the definition of multiplication.*

Universally. If all the products, when the number of factors is n, be equal, all the products, when the number of factors is $n + 1$, will likewise be equal: for those *of the same class* are equal, being produced of equal quantities multiplied by the same quantity: and to show that those *of different classes* are equal also, we need only take two products which differ in their two last factors, and have all the preceding ones according to the same order, and prove them to be equal. These two factors we will suppose to be represented by r and s, and the product of all the preceding ones by p; then the two products themselves will be represented by prs and psr, which are equal, *by case* 2:

1°. *Simple quantities are multiplied together by multiplying the coefficients one into the other, and to the product annexing the quantity which, according to the method of notation, expresses the product of the species; prefixing the sign + or —, according as the signs of the given quantities are like or unlike.*

Thus	$2a$	Also	$6ab$	And	$11adf$
mult. by	$3b$	mult. by	$5c$	mult. by	$7ab$
makes	$6ab$.	makes	$30abc$.	makes	$77aabdf$.

Thus, by way of illustration, $abcde$ will appear to be $= abced$, &c. For, the former of these being equal to every other product of the class, or termination e (by hypothesis and equal multiplication), and the latter equal to every other product of the class, or termination d; it is evident, therefore, that all the products of different classes, as well as of the same class, are mutually equal to each other.

So far relates to the first general observation: it remains to prove that $abcd \times pqrst$ is $= a \times b \times c \times d \times p \times q \times r \times s \times t$. In order to which, let $abcd$ be denoted by x, then will $abcd \times pqrst$ be denoted by $x \times pqrst$, or $pqrst \times x$ (by case 1), that is, by $p \times q \times r \times s \times t \times x$; which is equal to $x \times p \times q \times r \times s \times t$, or $a \times b \times c \times d \times p \times q \times r \times s \times t$, *by the preceding demonstration.*

The reason of rule 1 depends on these two general observations: for it is evident from hence, that $2a \times 3b$ (in the first example) is $= 2 \times a \times 3 \times b = 2 \times 3 \times a \times b = 6 \times a \times b = 6ab$: and, in the same manner, $11adf \times 7ab$ (in the third example) appears to be $= 11 \times a \times d \times f \times 7 \times a \times b = 11 \times 7 \times a \times a \times b \times d \times f = 77 \times aabdf = 77\,aabdf$. But the grounds of the method of proceeding may be otherwise explained, thus: it has been observed that ab (according to the method of notation) defines the product of the species a, b (in the first example), therefore the product of a by $3b$, which must be three times as great (because the multiplier is here three times as great), In the preceding examples, all the products are *affirmative*, the quantities given to be multiplied being so; but, in those that follow, some are *affirmative*, and others *negative*, according to the different cases specified in the latter part of the rule; whereof the reasons will be explained hereafter.

$$\begin{array}{llllll}
\text{Mult.} & +\ 5a & \text{Mult.} & -\ 5a & \text{Mult.} & -\ 5a \\
\text{by} & -\ 6b & \text{by} & +\ 6b & \text{by} & -\ 6b \\
\hline
\text{Prod.} & -30ab. & \text{Prod.} & -30ab. & \text{Prod.} & +30ab.
\end{array}$$

$$\begin{array}{llll}
\text{Mult.} & +7\sqrt{ax} & \text{Mult.} & -7a\sqrt{aa+xx} \\
\text{by} & -5\sqrt{cy} & \text{by} & -6b\sqrt{aa-yy} \\
\hline
\text{Prod.} & -35\times\sqrt{ax}\times\sqrt{cy}. & \text{Prod.} & +42ab\times\sqrt{aa+xx}\times\sqrt{aa-yy}
\end{array}$$

In the two last examples, and all others where radical quantities are concerned, every such quantity may be considered, and treated in all respects as a simple quantity, expressed by a single letter; since it is not the form of the expression, but the value of the quantity that is here regarded.

2°. *A fraction is multiplied by multiplying the numerator thereof by the given multiplier, and making the product a numerator to the given denominator.*

Thus $\frac{a}{b}\times c$ makes $\frac{ac}{b}$; also $\frac{3ac}{b}\times 2ad$ makes $\frac{6aacd}{b}$;

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will be truly defined by $3ab$, or ab taken three times: but since the product of a by $3b$ appears to be $3ab$, it is plain that the product of $2a$ by $3b$ must be twice as great as that of a by $3b$, and therefore will be truly expressed by $6ab$. Thus, also, the product of the species ab and c, in the second example, being abc by bare notation, it is evident that the product of $6ab$ by c will be truly defined by $6abc$, or acb six times taken, and consequently the product of $6ab$ and $5c$, by $30abc$, or $6abc$ taken five times, the multiplier here being five times as great.

The reason of rule 2° may be thus demonstrated: let the numerator of any proposed fraction be denoted by A, likewise $\frac{2ab}{c+d} \times 7\sqrt{ax}$ makes $\frac{14ab\sqrt{ax}}{c+d}$; lastly, $\frac{5ab}{\sqrt{aa+xx}} \times 2ab$ makes $\frac{10a^2b^2}{\sqrt{aa+xx}}$.

3°. *Fractions are multiplied into one another by multiplying the numerators together for a new numerator, and the denominators together for a new denominator.*

Thus, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$; $\frac{2ab}{3c} \times \frac{5ad}{3f} = \frac{10a^2bd}{9cf}$; $\frac{21xy}{\sqrt{az}} \times \frac{3a\sqrt{a}}{8b} = \frac{63axy\sqrt{a}}{8b\sqrt{az}}$; $\frac{-5a\sqrt{x}}{3bc} \times \frac{-2a}{b} = \frac{10a^2\sqrt{x}}{3bbc}$; and $\frac{3a\sqrt{xy}}{\sqrt{ab}} \times \frac{5b\sqrt{aa+xx}}{a+z} = \frac{15ab \times \sqrt{xy} \times \sqrt{aa+xx}}{\overline{a+z} \times \sqrt{ab}}$.

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the denominator by B, and the given multiplicator by C: then, I say, that $\frac{AC}{B}$ is equal to $\frac{A}{B} \times C$. For, since $\frac{AC}{B}$ denotes the quantity which arises by dividing AC by B, and $\frac{A}{B}$ the quantity which arises by dividing A by B, it is evident that the former of these two quantities must be C times as great as the latter, because the dividual is C times as great in the one case as in the other; and therefore must be equal to the latter C times taken, that is, $\frac{AC}{B}$ must be equal to $\frac{A}{B} \times C$, as was to be shown.

The reason of rule 3° will appear evident from the preceding demonstration of rule 2°. For, it being there proved that $\frac{A}{B} \times C$ is equal to $\frac{AC}{B}$, it is obvious that $\frac{A}{B} \times \frac{C}{D}$ can be only the D part of $\frac{AC}{B}$; be-

4°. *Surd quantities under the same radical sign are multiplied like rational quantities, only the product must stand under the same radical sign.*

Thus, $\sqrt{7} \times \sqrt{5} = \sqrt{35}$; $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$; $\sqrt[3]{7bc} \times \sqrt[3]{5ad} = \sqrt[3]{35abcd}$; $3\sqrt{ab} \times 5\sqrt{c} = 15\sqrt{abc}$; $2a\sqrt{2cy} \times 3b\sqrt{5ax}$ $(= 6ab \times \sqrt{2cy} \times \sqrt{5ax})$ $= 6ab\sqrt{10acxy}$; and $\frac{7ab}{5x}\sqrt{\frac{8x}{3a}} \times \frac{5c}{9d}\sqrt{\frac{13d}{2b}} =$ $\frac{35abc}{45dx}\sqrt{\frac{104dx}{6ab}}$.

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cause $\frac{C}{D}$, the multiplier here, is but the **D** part of the former multiplier **C**: but $\frac{AC}{BD}$ is also equal to the **D** part of the same $\frac{AC}{B}$; because its divisor is **D** times as great as that of $\frac{AC}{B}$: therefore these two quantities, $\frac{A}{B} \times \frac{C}{D}$ and $\frac{AC}{BD}$, being the same part of one and the same quantity, must necessarily be equal to each other; *which was to be proved.*

As to rule 4° for the multiplication of similar radical quantities, it may be explained thus: suppose $\sqrt{A}$ and $\sqrt{B}$ to represent the two given quantities to be multiplied together; let the former of them be denoted by a, and the latter by b, that is, let the quantities represented by a and b be such, that aa may be $=$ A, and $bb = B$; then the product of $\sqrt{A}$ by $\sqrt{B}$, or of a by b, will be expressed by ab, and its square by $ab \times ab$: but $ab \times ab$ is $= a \times b \times a \times b = aa \times bb$ (by the general observations premised at the beginning of this section); whence the square of the product is likewise truly expressed by $aa \times bb$, or its equal $A \times B$; and consequently the product itself by $\sqrt{A \times B}$, that is, by the quantity which, being multiplied into itself, produces $A \times B$.

5°. *Powers, or roots, of the same quantity are multiplied together by adding their exponents:* but the exponents here understood are those defined in p. 5, where roots are represented as fractional powers.

Thus, $x^2 \times x^3$ is $= x^5$; $\overline{a+z}\rceil^3 \times \overline{a+z}\rceil^5 = \overline{a+z}\rceil^8$; $x^2 \times x^{\frac{1}{2}} = x^{2+\frac{1}{2}} = x^{\frac{5}{2}}$; and $x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^1 = x$; also $\overline{aa+zz}\rceil^{\frac{2}{3}} \times \overline{aa+zz}\rceil^{\frac{1}{3}}$ is $= \overline{aa+zz}\rceil^1 = aa+zz$; and $\overline{c+y}\rceil^{\frac{1}{2}} \times \overline{c+y}\rceil^{\frac{1}{3}} = \overline{c+y}\rceil^{\frac{1}{2}+\frac{1}{3}} = \overline{c+y}\rceil^{\frac{5}{6}}$*.

In the same manner, the product of $\sqrt[3]{\text{A}} \times \sqrt[3]{\text{B}}$ will appear to be $\sqrt[3]{\text{AB}}$: for if $\sqrt[3]{\text{A}}$ be denoted by a, and $\sqrt[3]{\text{B}}$ by b; or, which is the same thing, if $aaa = \text{A}$, and $bbb = \text{B}$; then will $\sqrt[3]{\text{A}} \times \sqrt[3]{\text{B}} = a \times b$ (or ab) and its cube $= ab \times ab \times ab = aaa \times bbb = \text{AB}$ (by the aforesaid observations); whence the product itself will evidently be expressed by $\sqrt[3]{\text{AB}}$.

* The grounds of these operations may be thus explained. First, when the exponents are whole numbers, as in example 1, the demonstration is obvious, from the general observations premised at the beginning of the section: for, by what is there shown, $x^2 \times x^3$, or $xx \times xxx$ is $= x \times x \times x \times x \times x = x^5$ (*by notation*). But in the last example, where the exponents are fractions, let $\overline{c+y}\rceil^{\frac{1}{6}}$ be represented by x; that is, let the quantity x be such, that $x \times x \times x \times x \times x \times x$, or x^6 may be equal to $c+y$; so shall $\overline{c+y}\rceil^{\frac{1}{2}}$ be expressed by x^3; because, by what has been already shown, $x^3 \times x^3$ is $= x^6$: and in the same manner will $\overline{c+y}\rceil^{\frac{1}{3}}$ be expressed by x^2; because $x^2 \times x^2 \times x^2$ is likewise $= x^6$. Therefore $\overline{c+y}\rceil^{\frac{1}{2}} \times \overline{c+y}\rceil^{\frac{1}{3}}$ is $= x^3 \times x^2 = x^5 =$ the fifth power of $\overline{c+y}\rceil^{\frac{1}{6}}$; which is $\overline{c+y}\rceil^{\frac{5}{6}}$, *by notation.*

6°. *A compound quantity is multiplied by a simple one, by multiplying every term of the multiplicand by the multiplier.*

Thus $a + 2b - 3c$ Also $a2 - 5a\sqrt{x + 7b}$
mult. by $3a$ mult. by $8c$
makes $3a^2 + 6ab - 9ac$; makes $8a^2c - 40ac\sqrt{x + 56bc}$;

And $5a^2 - 8ab + 6ac - 7bc + 12b^2 - 9c^2$
mult. by $3abc$
makes $15a^3bc - 24a^2b^2c + 18a^2bc^2 - 21ab^2c^2 + 36ab^3c -- 27abc^3$.

To explain the reason of the two last rules, let it be first proposed to multiply any compound quantity, as $a + b - c - d$, by any simple quantity f; and, I say, the product will be $af + bf - cf - df$. For the product of the affirmative terms, $a + b$, will be $af + bf$, because to multiply one quantity by another is to take the multiplicand as many times as there are units in the multiplier, and to take the whole multiplicand $(a + b)$ any number of times (f), is the same as to take all its parts (a, b) the same number of times, and add them together. Moreover, seeing $a + b - c - d$ denotes the excess of the affirmative terms (a and b) above the negative ones (c and d), therefore, to multiply $a + b - c - d$ by f, is only to take the said excess f times; but f times the excess of any quantity above another is, manifestly, equal to f times the former quantity, *minus* f times the latter; but f times the former is, here, equal to $af + bf$ (by what has been already shown), and f times the latter, for the same reason, will be equal to $cf + df$, and therefore the product of $a + b - c - d$ by f is equal to $af + bf - cf - df$; as was to be proved. Hence it appears, that a compound quantity is multiplied by a simple affirmative quantity, by multiplying every term of the former by the latter, and connecting the terms thence arising with the signs of the multiplicand.

7°. *Compound quantities are multiplied into one another by multiplying every term of the multiplicand by each term of the multiplier, successively, and connecting the several products thus arising with the signs of the multiplicand, if the multiplying term be affirmative, but with contrary signs, if negative.*

Thus the product of	$5a + 3x$
multiplied by	$3a + 2x$
will be	$\left\{\begin{array}{l} 15aa + 9ax \\ \qquad\quad + 10ax + 6xx \end{array}\right\}$
which contracted by uniting the like terms, is	$\left\{ 15aa + 19ax + 6xx. \right.$

But to prove that the method also holds good when both the quantities are compound ones, let it be now proposed to multiply A — B by C — D; then, I say, the product will be truly expressed by AC — BC — AD + BD. For, it has been already observed, that to multiply one quantity by another is to take the multiplicand as many times as there are units in the multiplier; and, therefore, to mutiply A — B by C — D is only to take A — B as many times as there are units in C — D: Now (according to the method of multiplying compound quantities) I first take A — B, C times (or multiply by C), and the quantity thence arising will be AC — BC, *by what is demonstrated above.* But I was to have taken A — B only C — D times; therefore, by this first operation, I have taken it D times too much; whence, to have the true product, I ought to deduct D times A — B from AC — BC, the quantity thus found; but D times A — B, *by what is already proved*, is equal to AD — BD; which subtracted from AC — BC, or written down with its signs changed, gives the true product, AC — BC — AD + BD, *as was to be demonstrated.* And, *universally*, if the sign of any proposed term of the multiplier, in any case whatever, be affirmative, it is easy to conceive that the required product will be greater than it would be if there

Likewise the product

$$\begin{array}{ll} \text{of} & a^3 + a^2b + ab^2 + b^3 \\ \text{by} & a - b \\ \hline \text{is} & \left\{ \begin{array}{l} a^4 + a^3b + a^2b^2 + ab^3 \\ \quad - a^3b - a^2b^2 - ab^3 - b^4 \end{array} \right\} \end{array}$$

Which, by striking out the terms that destroy one another, becomes $a^4 - b^4$.

were no such term, by the product of that term into the whole multiplicand; and therefore it is, that this product is to be added, or written down with its proper signs, which are proved above to be those of the multiplicand. But if, on the contrary, the sign of the term by which you multiply be negative; then, as the required product must be less than it would be if there were no such term, by the product of that term into the whole multiplicand, this product, it is manifest, ought to be subtracted, or written down with contrary signs.

Hence is derived the common rule, *that like signs produce +, and unlike signs, —.*

For, first, if the signs of both the quantities, or terms, to be multiplied be affirmative, and therefore *like*, it is plain that the sign of the product must likewise be affirmative.

Secondly, also, if the signs of both quantities be negative, and therefore still *like*, that of the product will be affirmative, *because contrary to that of the multiplicand, by what has been just now proved.*

Thirdly, but if the sign of the multiplicand be affirmative, and that of the multiplier negative, and therefore *unlike*, the sign of the product will be negative, *because contrary to that of the multiplicand.*

Lastly, if the sign of the multiplicand be negative, and that of the multiplier affirmative, and therefore still *unlike*, the sign of the product will be negative, *because the same with that of the multiplicand.*

And these four are all the cases that can possibly happen with regard to the variation of signs.

Other examples in multiplication, for the learner's exercise, may be as follows; from which he may, if he please, proceed directly to division, by passing over the intervening scholium.

$$\begin{array}{lllllll}
\text{1. Multiply} & x^2 & +xy & +y^2 & & & \\
\text{by} & x^2 & -xy & +y^2 & & & \\ \hline
 & x^4 & +x^3y & +x^2y^2 & & & \\
 & & -x^3y & -x^2y^2 & -xy^3 & & \\
 & & & +x^2y^2 & +xy^3 & +y^4 & \\ \hline
\text{product} & x^4 & * & +x^2y^2 & * & +y^4. &
\end{array}$$

$$\begin{array}{llllll}
\text{2. Multiply} & 2a^2 & -3ax & +4x^2 & & \\
\text{by} & 5a^2 & -6ax & -2x^2 & & \\ \hline
 & 10a^4 & -15a^3x & +20a^2x^2 & & \\
 & & -12a^3x & +18a^2x^2 & -24ax^3 & \\
 & & & -4a^2x^2 & +6ax^3 & -8x^4 \\ \hline
\text{product} & 10a^4 & -27a^3x & +34a^2x^2 & -18ax^3 & -8x^4.
\end{array}$$

$$\begin{array}{lllllll}
\text{3. Multiply} & 3a & -2b & +2c & & & \\
\text{by} & 2a & -4b & +5c & & & \\ \hline
 & 6aa & -4ab & +4ac & & & \\
 & & -12ab & +8bb & -8bc & & \\
 & & & +15ac & -10bc & +10cc & \\ \hline
\text{product} & 6aa & -16ab & +19ac & +8bb & -18bc & +10cc.
\end{array}$$

$$\begin{array}{lllllll}
\text{4. Multiply} & a^3 & -3a^2b & +3ab^2 & -b^3 & & \\
\text{by} & a^2 & -2ab & +b^2 & & & \\ \hline
 & a^5 & -3a^4b & +3a^3b^2 & -a^2b^3 & & \\
 & & -2a^4b & +6a^3b^2 & -6a^2b^3 & +2ab^4 & \\
 & & & +a^3b^2 & -3a^2b^3 & +3ab^4 & -b^5 \\ \hline
\text{product} & a^5 & -5a^4b & +10a^3b^2 & -10a^2b^3 & +5ab^4 & -b^5
\end{array}$$

SCHOLIUM.

The manner of proceeding, in referring the reasons of the different cases of the signs to the multiplication of compound quantities, may perhaps be looked upon as indirect, and contrary to good method; according to which, it may be thought that these reasons ought to have been given before, along with the rules for simple quantities, as this is the way that almost all authors on the subject have followed.

But, however indirect the method here pursued may seem, it appears to me the most clear and rational; and I believe it will be found very difficult, if not impossible, without explaining the rules for compound quantities first, to give a learner a *distinct* idea how the product of two simple quantities with negative signs, such as $-b$ and $-c$, ought to be expressed, when they stand alone, independent of all other quantities: and I cannot help thinking farther, that the difficulties about the signs, so generally complained of by beginners, have been much more owing to the manner of explaining them, this way, than to any real intricacy in the subject itself; nor will this opinion, perhaps, appear ill grounded, if it be considered that both $-a$ and $-b$, as they stand here independently, are as much impossible in one sense, as the imaginary surd quantities $\sqrt{-b}$ and $\sqrt{-c}$; since the sign $-$, according to the established rules of notation, shows that the quantity to which it is prefixed is to be subtracted; but, to subtract something from nothing is impossible, and the notion or supposition of a quantity less than nothing, absurd and shocking to the imagination: and, certainly, if the matter be viewed in this light, it would be very ridiculous to pretend to prove, by any show of reasoning, what the product of $-b$ by $-c$, or of $\sqrt{-b}$ by $\sqrt{-c}$, must be, when we can have no idea of the value of the quantities to be multiplied. If, indeed, we were to look upon $-b$ and $-c$ as *real* quantities, the same as represented to the mind by b and c (which cannot be done consistently, in pure algebra, where magnitude only is regarded), we might then attempt to explain the matter in the same manner that some others have done; from the consideration, that, as the sign $-$ is opposite in its nature to the sign $+$, it ought therefore to have, in all operations, an opposite effect; and, consequently, that as the product, when the sign + is prefixed to the multiplier, is to be added, so, on the contrary, the product, when the sign — is prefixed, ought to be subtracted.

But this way of arguing, however reasonable it may appear, seems to carry but very little of science in it, and to fall greatly short of the evidence and conviction of a demonstration: nay, it even clashes with first principles, and the more established rules of notation; according to which the signs + and — are relative only to the magnitudes of quantities, as composed of different terms or members, and not to any future operations to be performed by them: besides, when we are told that the product arising from a negative multiplier is to be subtracted, we are not told what it is to be subtracted from; nor is there any thing from whence it can be subtracted, when negative quantities are independently considered. And, farther, to reason about opposite effects, and recur to sensible objects and popular considerations, such as debtor and creditor, *&c.* in order to demonstrate the principles of a science whose object is abstract number, appears to me not well suited to the nature of science, and to derogate from the dignity of the subject.

It must be allowed, that, in the application of algebra to different branches of mixed mathematics, where the consideration of opposite qualities, effects, or positions can have place, the usual methods have a better foundation; and the conception of a quantity absolutely negative becomes less difficult. Thus, for example, a line may be conceived to be produced out, both ways, from any point assigned; and the part on the one side of that point being taken as *positive*, the other will be *negative*. But the case is not the same in abstract number, whereof the beginning is fixed in the nature of things, from whence we can proceed only one way.

There can, therefore, be no such things as negative numbers, or quantities absolutely negative, in pure algebra, whose object is number, and where every multiplication, division, *&c.* is a multiplication, division, &c. of numbers, even in the application thereof; for, when we reason upon the quantities *themselves*, and not upon the *numbers* expressing the measures of them, the process becomes *purely geometrical*, whatever symbols may be used therein, from the algebraic notation; which can be of no other use here than to abbreviate the work.

However, after all, it may be necessary to show upon what kind of evidence the multiplication of negative and imaginary quantities is grounded, as these sometimes occur, in the resolution of problems: in order to which it will be requisite to observe, that, as all our reasoning regards *real*, *positive* quantities, so the algebraic expressions, whereby such quantities are exhibited, must likewise be real and positive. But, when the problem is brought to an equation, the case may indeed be otherwise; for, in ordering the equation, so much may be taken away from both sides thereof, as to leave the remaining quantities negative; and then it is, chiefly, that the multiplication by quantities absolutely negative takes place.

Thus, if there were given the equation $a - \frac{x}{b} = c$, in order to find x; then, by subtracting the quantity a from each side thereof, we shall have $-\frac{x}{b} = c - a$; which multiplied by $-b$, *according to the general rule*, gives $x = -cb + ab$; that is, $-\frac{x}{b}$ by $-b$ will give $+x$; c by $-b$, $-cb$; and $-a$ by $-b$, $+ab$; which appear to be true; because the products being thus expressed, the same conclusion is derived, as if both sides of the original equation had been first increased by $\frac{x}{b} - c$, and then multiplied by b; where both the multiplier and multiplicand are real, affirmative quantities, and where the whole operation is, therefore, capable of a clear and strict demonstration: but then it is not in consequence of any reasoning I am capable of forming about $-\frac{x}{b}$ and $-b$, or about $+c$ and $-b$, considered independently, that I can be *certain* that their product ought to be expressed in that manner.

So, likewise, if there were given the equation $a - \frac{x^2}{b} = c$; by transposing a, and taking the square root on both sides, we shall have $\sqrt{-\frac{x^2}{b}} = \sqrt{c-a}$; and this multiplied by $\sqrt{-b}$, will give $\sqrt{x^2}$ (or x) $= \sqrt{-cb+ab}$: which also appears to be true, because the result, this way, comes out exactly the same as if the operations for finding x had been performed altogether by *real* quantities: but, notwithstanding this, it is not from any reasoning that I can form about the multiplication of the *imaginary* quantities $\sqrt{-\frac{x^2}{b}}$ and $\sqrt{-b}$, &c. considered independently, that I can prove their product ought to be so expressed; for it would be very absurd to pretend to demonstrate what the product of two expressions must be, which are impossible in themselves, and of whose values we can form no idea. It indeed seems reasonable, that the known rules for the signs, as they are proved to hold good in all cases whatever, where it is possible to form a demonstration, should also answer *here:* but the strongest evidence we can have of the truth and certainty of conclusions derived by means of negative and imaginary quantities, is the exact and constant agreement of such conclusions with those determined from more demonstrable methods, wherein no such quantities have place.

In the foregoing considerations, the negative quanties $-b$, $-c$, &c. have been represented, in some cases, as a kind of imaginary or impossible quantities; it may not, therefore, be improper to remark here, that such imaginary quantities serve, many times, as much to discover the impossibility of a problem, as imaginary surd quantities: for it is plain, that, in all questions relating to abstract numbers, or such wherein magnitude only is regarded, and where no consideration of position, or contrary values, can have place; I say, in all such cases, it is plain that the solution will be altogether as impossible, when the conclusion comes out a negative quantity, as if it were actually affected with an imaginary surd; since, in the one case, it is required that a number should be actually less than nothing; and, in the other, that the double-rectangle of two numbers should be greater than the sum of their squares; both which are equally impossible: but, as an instance of the impossibility of some sort of questions, when the conclusion comes out negative, let there be given, in a right-angled triangle, the sum of the hypothenuse and perpendicular $= a$, and the base $= b$, to find the perpendicular; then, by what shall hereafter be shown in its proper place, the answer will come out $\frac{a^2 - b^2}{2a}$, and is possible, or impossible, according as the quantity $\frac{a^2 - b^2}{2a}$ is affirmative or negative, or as a is greater or less than b; which will manifestly appear from a bare contemplation of the problem: and the same thing might be instanced in a variety of other examples.

SECTION V.

Of Division.

DIVISION in species, as in numbers, is the converse of multiplication, and is comprehended in the seven following cases.

1°. *When one simple quantity is to be divided by another, and all the factors of the divisor are also found in the dividend, let those factors be all cast off or expunged, and then the remaining factors of the dividend, joined together, will express the quotient sought.* But it is to be observed, that, both here and in the succeeding cases, the same rule is to be regarded, in relation to the signs, as in multiplication, *viz.* that *like signs give* $+$, *and unlike*, $-$. It may also be proper to observe, that, when any quantity is to be divided by itself, or an equal quantity, the quotient will be expressed by a unit, or 1.

Thus $a \div a$ gives 1; and $2ab \div 2ab$ gives 1;
moreover, $3abcd \div ac$ gives $3bd$;
and $16bc \div 8b$ gives $2c$: for here the dividend, by resolving its coefficient into two factors, becomes $2 \times 8 \times b \times c$; from whence casting off 8 and b, those common to the divisor, we have $2 \times c$, or $2c$. In the same manner, by resolving or dividing the coefficient of the dividend by that of the divisor, the quotient will be had in other cases: Thus, $20abc$, divided by $4c$, gives $5ab$; and $-\ 51ab\sqrt{xy} \times \sqrt{xx + yy}$, divided by $-\ 17a\sqrt{xy}$, gives $+\ 3b\sqrt{xx + yy}$.

2°. *But if all the factors of the divisor be not found in the dividend, cast off those (if any such there be) that are common to both, and write down the remaining factors of the divisor, joined together, as a denominator to those of the dividend; so shall the fraction thus arising express the quotient sought.* But if, by proceeding thus, all the factors in the dividend should happen to go off, or vanish, then a unit will be the numerator of the fraction required.

Thus, abc, divided by bcd, gives $\frac{a}{d}$:

And $16a^2bx^3$, divided by $8abcx^2$, gives $\frac{2ax}{c}$:

The first rule, given above, being exactly the converse of rule 1° in the preceding section, requires no other demonstration than is there given. The second rule (as well as those that follow hereafter upon fractions) depends on this principle, that, as many times as any one proposed quantity is contained in another, just so many times is the half, third, fourth, or any other assigned part of the former, contained in the half, third, fourth, or other corresponding part of the latter; and

Likewise, $27ab\sqrt{xy}$, divided by $9a^2\sqrt{xy}$, gives $\frac{3b}{a}$:

And $8ab\sqrt{ay}$, divided by $16a^2b\sqrt{ay}$, gives $\frac{1}{2a}$.

3°. *One fraction is divided by another, by multiplying the denominator of the divisor into the numerator of the dividend for a new numerator, and the numerator of the divisor into the denominator of the dividend for a new denominator.*

Thus, $\frac{a}{b}$, divided by $\frac{c}{d}$, gives $\frac{ad}{bc}$:

Also, $\frac{5ax}{3c}$, divided by $\frac{6bc}{7d}$, gives $\frac{35adx}{18bcc}$:

And $\frac{6a^2b}{5x}$, divided by $\frac{5ab^2}{3x}$, gives $\frac{18a^2bx}{25ab^2x}$.

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just so many times, likewise, is the double, triple, quadruple, or any other assigned multiple of the former, contained in the double, triple, quadruple, or other corresponding multiple of the latter. The demonstration of this principle (though it may be thought too obvious to need one) may be thus: let A and B represent any two proposed quantities, and AC and BC their equimultiples (or, let AC and BC be the two quantities, and A and B their like parts): I say, then, that $\frac{AC}{BC} = \frac{A}{B}$: for the multiple of $\frac{AC}{BC}$ by BC is manifestly $= AC$; and $\frac{A}{B} \times BC$, the multiple of $\frac{A}{B}$ by the same BC, is $= \frac{A \times BC}{B}$ (*by rule 2 in multiplication*) $= \frac{ACB}{B}$ (*vid. p.* 14 *and* 15) $= AC$: therefore, seeing the equimultiples of the two proposed quantities are the same, the quantities themselves must necessarily be equal.

The second rule, given above, is nothing more than a bare application of the principle here demonstrated;

But, in cases like this last, where the two numerators, or the denominators, have factors common to both, the conclusion will become more neat by first casting off such common factors.

Thus casting away ab out of the two numerators, and x out of both the denominators, we have $\frac{6a}{5}$ to be divided by $\frac{5b}{3}$; whereof the quotient is $\frac{18a}{25b}$: in the same manner, $\frac{12ac^3}{10bb} \div \frac{4acx}{5bd}$, or $\frac{3c^2}{2b} \div \frac{x}{d}$, gives $\frac{3c^2d}{2bx}$; and $\frac{6a\sqrt{xy}}{5c} \div \frac{7a^2\sqrt{xy}}{10bc}$, or $\frac{6}{1} \div \frac{7a}{2b}$, gives $\frac{12b}{7a}$.

When either the divisor or the dividend is a *whole quantity*, instead of a fraction, it may be reduced to the form of a fraction, by writing a unit or 1 under it.

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since, by casting off the factors common to the dividend and divisor (as directed in the rule), it is plain that we take like parts of those quantities: therefore the quotient arising by dividing the one part by the other will be the same as that arising by dividing one whole by the other.

As to rule 3°, wherein it is asserted that $\frac{A}{B} \div \frac{C}{D} = \frac{AD}{BC}$, it is evident that AD and BC are equimultiples of the given quantities $\frac{A}{B}$ and $\frac{C}{D}$; because $\frac{A}{B} \times BD$ is (*by rule* 2° *in multiplication*) $= \frac{ABD}{B} = AD$, and $\frac{C}{D} \times BD = \frac{CBD}{D} = CB$: whence it follows that the quotient of $\frac{A}{B}$ divided by $\frac{C}{D}$ will be the same with that of AD divided by BC; which, by notation, is $\frac{AD}{BC}$, as was to be shown. The grounds of the note subjoined to this rule are these: by casting away all factors common

Thus, $\frac{12ab}{5c}$, divided by $7d$ (or $\frac{7d}{1}$), gives $\frac{12ab}{35cd}$;

And $5a^2b$ (or $\frac{5a^2b}{1}$), divided by $\frac{9x^2}{3y}$, gives $\frac{15a^2by}{9xx}$.

4°. *Surd quantities, under the same radical sign, are divided by one another like rational quantities, only the quotient must stand under the given radical sign.*

Thus, the quotient of $\sqrt{ab}$ by $\sqrt{b}$ is $\sqrt{a}$:

That of $\sqrt[3]{16xxy}$ by $\sqrt[3]{8xy}$ is $\sqrt[3]{2x}$:

That of $\sqrt{\frac{10abb}{3c}}$ by $\sqrt{\frac{5ab}{c}}$ is $\sqrt{\frac{10abbc}{15abc}}$, or $\sqrt{\frac{2b}{3}}$.

And that of $6ab\sqrt{10acxy}$ by $2a\sqrt{2cy}$ is $3b\sqrt{5ax}$.

5°. *Different powers, or roots, of the same quantity are divided one by another, by subtracting the exponent of the divisor from that of the dividend, and placing the remainder as an exponent to the quantity given.* But it must be observed, that the exponents here understood are those defined in p. 5; where all roots are represented as fractional powers. It will likewise be proper to remark further, that, when the exponent of the divisor is greater than that of the dividend, the quotient will have a negative exponent, or, which comes to the same thing, the result will be a fraction, whereof the numerator is a unit, and the denominator the same quantity with its exponent changed to an affirmative one.

Thus, x^5, divided by x^2, gives x^3:

And $\overline{a+z}|^7$, divided by $\overline{a+z}|^3$, gives $\overline{a+z}|^4$:

Likewise, $x^{\frac{1}{2}}$, divided by $x^{\frac{1}{4}}$, gives $x^{\frac{1}{4}}$:

---

to the two numerators, we take equal parts of the quantities; and, by throwing off the factors common to both denominators, we take equimultiples of those parts.

The two preceding rules, being nothing more than the converse of the 4th and 5th rules in multiplication, are demonstrated in them: though, perhaps, the case in rule 5, where the exponent comes out negative, may stand in need of a more particular explanation. Accord-

Moreover, $\overline{c+y}|^{\frac{1}{2}}$, divided by $\overline{c+y}|^{\frac{1}{3}}$, gives $\overline{c+y}|^{\frac{1}{6}}$:
Lastly, x^3, divided by x^5, gives x^{-2}, or $\frac{1}{x^2}$. or —X2

6°. *A compound quantity is divided by a simple one by dividing every term thereof by the given divisor.*

Thus, $3ab)\ 3abc + 12abx - 9aab\ (c + 4x - 3a$:
Also, $-5ac)15a^2bc - 12acy^2 + 5ad^2(-3ab + \frac{12yy}{5} - \frac{dd}{c}$;
and so of others.

7°. *But if the divisor, as well as the dividend, be a compound quantity, let the terms of both quantities be disposed in order, according to the dimensions of some letter in them, as shall be judged most expedient, so that those terms may stand first wherein the highest power of that letter is involved, and those next where the next highest power is involved, and so on: this being done, seek how many times the first term of the divisor is contained in the first term of the dividend, which, when found, place in the quotient, as in division in vulgar arithmetic, and then multiply the whole divisor thereby, subtracting the product from the respective terms of the dividend; to the remainder bring down, with their proper signs, as many of the next following terms of the dividend as are requisite for the next operation; seeking again how often the first term of the divisor is contained in the first term of the remainder, which also write down in your quotient, and proceed as before, repeating the operation till all the terms of the dividend are exhausted, and you have nothing remaining.*

---

ing to the said rule, the quotient of x^3 divided by x^5 was asserted to be x^{-2}, or $\frac{1}{x^2}$. Now, that this is the true value is evident; because 1 and x^2 being like parts of x^3 and x^5 (which arise by dividing by x^3), their quotient will consequently be the same with that of the quantities themselves.

Thus, if it were required to divide $a^3 + 5a^2x + 5ax^2 + x^3$ by $a + x$ (where the several terms are disposed according to the dimensions of the letter a), I first write down the divisor and dividend, in the manner below, with a crooked line between them, as in the division of whole numbers; then, I say, how often is a contained in a^3, or what is the quotient of a^3 by a; the answer is a^2, which I write down in the quotient, and multiply the whole divisor, $a + x$, thereby, and there arises $a^3 + a^2x$; which subtracted from the two first terms of the dividend, leaves $4a^2x$; to this remainder I bring down $+ 5ax^2$, the next term of the dividend, and then seek again how many times a is contained in $4a^2x$; the answer is $4ax$, which I also put down in the quotient, and by it multiply the whole divisor, and there arises $4a^2x + 4ax^2$, which subtracted from $4a^2x + 5ax^2$ leaves ax^2, to which I bring down x^3, the last term of the dividend, and seek how many times a is contained in ax^2, which I find to be x^2; this I therefore also write down in the quotient, and by it multiply the whole divisor; and then, having subtracted the product from $ax^2 + x^3$, find there is nothing remains; whence I conclude, that the required quotient is truly expressed by $a^2 + 4ax + x^2$. See the operation.

$$\begin{array}{rllllll}
a + x) & a^3 & + 5a^2x & + 5ax^2 & + x^3 & (a^2 + 4ax + x^2 \\
 & a^3 & + a^2x & & & \\
\cline{2-4}
 & & 4a^2x & + 5ax^2 & & \\
 & & 4a^2x & + 4ax^2 & & \\
\cline{3-5}
 & & & ax^2 & + x^3 & \\
 & & & ax^2 & + x^3 & \\
\cline{4-5}
 & & & 0 & 0 &
\end{array}$$

In the same manner, if it be proposed to divide $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$ by $a^2 - 2ax + x^2$, the quotient will come out $a^3 - 3a^2x + 3ax^2 - x^3$, as will appear from the process.

$$\begin{array}{l}
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -3a^2x+3ax^2-x^3 \\
a^2-2ax+x^2)\,a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5\,(a^3 \\
\qquad\qquad\quad \underline{a^5-2a^4x+\quad a^3x^2\qquad\quad} \\
\qquad\qquad\qquad -3a^4x+9a^3x^2-10a^2x^3 \\
\qquad\qquad\qquad \underline{-3a^4x+6a^3x^2-3a^2x^3\qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad +3a^3x^2-7a^2x^3+5ax^4 \\
\qquad\qquad\qquad\qquad\qquad \underline{+3a^3x^2-6a^2x^3+3ax^4\qquad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\;a^2x^3+2ax^4-x^5 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-\;a^2x^3+2ax^4-x^5} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0\qquad\quad 0\qquad\; 0
\end{array}$$

So, likewise, if a^5-x^5 be divided by $a-x$, the quotient will be $a^4+a^3x+a^2x^5+ax^2+x^4$; as by the work will appear.

$$\begin{array}{l}
a-x)\,a^5-x^5\,(a^4+a^3x+a^2x^2+ax^3+x^4 \\
\qquad\quad \underline{a^5-a^4x\quad} \\
\qquad\qquad\quad a^4x-x^5 \\
\qquad\qquad\quad \underline{a^4x-a^3x^2\quad} \\
\qquad\qquad\qquad\quad a^3x^2-x^5 \\
\qquad\qquad\qquad\quad \underline{a^3x^2-a^2x^3\quad} \\
\qquad\qquad\qquad\qquad\quad a^2x^3-x^5 \\
\qquad\qquad\qquad\qquad\quad \underline{a^2x^3-ax^4\quad} \\
\qquad\qquad\qquad\qquad\qquad\quad ax^4-x^5 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{ax^4-x^5} \\
\qquad\qquad\qquad\qquad\qquad\qquad 0\quad\; 0
\end{array}$$

Moreover, if it were required to divide $a^6-3a^4x^2+3a^2x^4-x^6$ by $a^3-3a^2x+3ax^2-x^3$, the process will stand thus:

$$\begin{array}{l}
a^3-3a^2x+\;\big)\,a^6-3a^4x^2+3a^2x^4-x^6\,(a^3+3a^2x+3ax^2+x^3 \\
\quad 3ax^2-x^3\;\big)\,\underline{a^6-3a^5x+3a^4x^2-a^3x^3\qquad\qquad} \\
\qquad\qquad\qquad\quad +3a^5x-6a^4x^2+\;a^3x^3+3a^2x^4 \\
\qquad\qquad\qquad\quad \underline{+3a^5x-9a^4x^2+9a^3x^3-3a^2x^4\qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad\quad +3a^4x^2-8a^3x^3+6a^2x^4-x^6 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{+3a^4x^2-9a^3x^3+9a^2x^4-3ax^5\quad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad +a^3x^3-3a^2x^4+3ax^5-x^6 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{+a^3x^3-3a^2x^4+3ax^5-x^6} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0\qquad\quad 0\qquad\quad 0\qquad 0
\end{array}$$

$a^3 + 2a^2b + ab^2$
$+ a^2b + 2ab^2 + b^3$
$a^3 + 3a^2b + 3ab^2 + b^3$ the cube, or third power.
$a + b$
$a^4 + 3a^3b + 3a^2b^2 + ab^3$
$+ a^3b + 3a^2b^2 + 3ab^3 + b^4$
$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, the fourth power.
$a + b$
$a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4$
$+ a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5$
$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, the 5th power.
$a + b$
$a^6 + 5a^5b + 10a^4b^2 + 10a^3b^3 + 5a^2b^4 + ab^5$
$+ a^5b + 5a^4b^2 + 10a^3b^3 + 10a^2b^4 + 5ab^5 + b^6$
$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$, the 6th or required power of $a + b$.

So, likewise, if it be required to involve or raise $a - b$ to the sixth power, the process will stand thus:

$a - b$
$a - b$
$a^2 - ab$
$- ab + b^2$
$a^2 - 2ab + b^2$
$a^2 - 2ab + b^2$, second power.
$a - b$
$a^3 - 2a^2b + ab^2$
$- a^2b + 2ab^2 - b^3$
$a^3 - 3a^2b + 3ab^2 - b^3$, third power.
$a - b$
$a^4 - 3a^3b + 3a^2b^2 - ab^3$
$- a^3b + 3a^2b^2 - 3ab^3 + b^4$
$a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$, fourth power.
$a - b$
$a^5 - 4a^4b + 6a^3b^2 - 4a^2b^3 + ab^4$
$- a^4b + 4a^3b^2 - 6a^2b^3 + 4ab^4 - b^5$
$a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$, fifth power.
$a - b$

$$\begin{array}{l} a^6 - 5a^5b + 10a^4b^2 - 10a^3b^3 + 5a^2b^4 - ab^5 \\ \quad - a^5b + 5a^4b^2 - 10a^3b^3 + 10a^2b^4 - 5ab^5 + b^6 \\ \hline a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6 \end{array}$$

the sixth power of $a - b$; and so of any other.

But there is a rule, or theorem, given by Sir *Isaac Newton*, demonstrated hereafter, whereby any power of a binomial $a + b$, or $a - b$, may be expressed in simple terms, without the trouble of those tedious multiplications required in the preceding operations; which is thus:

Let n denote any number at pleasure; then the nth power of $a + b$ will be $a^n + na^{\overline{n-1}}b + \frac{n \cdot n - 1}{1 \cdot 2} \cdot a^{\overline{n-2}}b^2 + \frac{n \cdot n - 1 \cdot n - 2}{1 \cdot 2 \cdot 3} \cdot a^{\overline{n-3}}b^3 + \frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot a^{\overline{n-4}}b^4 + \frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3 \cdot n - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot a^{\overline{n-5}}b^5$, &c.

And the nth power of $a - b$ will be expressed in the very same manner, only the signs of the second, fourth, sixth, *&c.* terms, where the odd powers of b are involved, must be negative.

An example or two will show the use of this general theorem.

First, then, let it be required to raise $a + b$ to the third power. Here n, the index of the proposed power, being 3, the first term, a^n, of the general expression, is equal to a^3; the second $na^{\overline{n-1}}b = 3a^2b$; the third $\frac{n \cdot n - 1}{1 \cdot 2} \cdot a^{\overline{n-2}}b^2 = 3ab^2$; the fourth $\frac{n \cdot n - 1 \cdot n - 2}{1 \cdot 2 \cdot 3} \cdot a^{\overline{n-3}}b^3 = b^3$; and the fifth $\frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot a^{\overline{n-4}}b^4$, &c. = nothing*. Therefore the third power of $a + b$ is truly expressed by $a^3 + 3a^2b + 3ab^2 + b^3$.

Again, let it be required to raise $a + b$ to the sixth power; in which case the index, n, being 6, we shall, by proceeding as in the last example, have $a^n = a^6$,

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* For one of the factors, $\frac{n-3}{4} = 0$. Ed.

$na^{n-1}b = 6a^5b$, $\frac{n \,.\, n-1}{1 \,.\, 2} \,.\, a^{n-2}b^2 = 15a^4b^2$, &c. and consequently, $\overline{a+b}|^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$; being the very same as was above determined by continual multiplication.

Lastly, let it be proposed to involve $cc + xy$ to the fourth power.

Here a must stand for cc, b for xy, and n for 4; then, by substituting these values, instead of a, b, and n, the general expression will become $c^8 + 4c^6xy + 6c^4x^2y^2 + 4c^2x^3y^3 + x^4y^4$, the true value sought.

From the preceding operations it may be observed, that the unciæ, or coefficients, increase till the indices of the two letters a and b become equal, or change values, and then return, or decrease again, according to the same order: therefore we need only find the coefficients of the first half of the terms in this manner, since from these the rest are given.

SECTION VII.

Of Evolution.

// *EVOLUTION, or the extraction of roots, being directly the contrary of involution, or raising of powers, is performed by converse operations,* viz. *by the division of indices, as involution was by their multiplication.*

Thus the square root of x^6, by dividing the exponent by 2, is found to be x^3; and the cube root of x^6, by dividing the exponent by 3, appears to be x^2: moreover, the biquadratic root of $\overline{a+x}|^8$ will be $\overline{a+x}|^2$; and the cube root of $\overline{aa+xx}|^2$ will be $\overline{aa+xx}|^{\frac{2}{3}}$.

In the same manner, if the quantity given be a fraction, or consist of several factors multiplied together, its root will be extracted, by extracting the root of each particular factor.

ebbers rule to find y root of a simple quantitie
tract the root of the coefficient for the numer
l part and ÷ the indices of the letters by the in
of the power and it wil give the root requi

Thus the square root of a^2b^2 will be ab; that of $\frac{a^2b^2}{c^2}$ will be $\frac{ab}{c}$; and that of $\frac{81 \times a^2 \times \overline{aa + xx}\rceil^4}{16 \times \overline{a - x}\rceil^2}$ will be $\frac{9a \times \overline{aa + xx}\rceil^2}{4 \times \overline{a - x}}$: moreover, the square root of $\overline{aa - xx}\rceil^{\frac{1}{2}}$ will be $\overline{aa - xx}\rceil^{\frac{1}{4}}$; its cube root $\overline{aa - xx}\rceil^{\frac{1}{6}}$; and its biquadratic root $\overline{aa - xx}\rceil^{\frac{1}{8}}$; and so of others. All which being nothing more than the converse of the operations in the preceding section, require no other demonstration than what is there given.

Evolution of compound quantities is performed by the following rule.

First, place the several terms, whereof the given quantity is composed, in order, according to the dimensions of some letter therein, as shall be judged most commodious; then let the root of the first term be found, and placed in the quotient; which term being subtracted, let the first term of the remainder be brought down, and divided by twice the first term of the quotient, or by three times its square, or four times its cube, &c. *according as the root to be extracted is a square, cubic, or biquadratic one,* &c. *and let the quantity thence arising be also written down in the quotient, and the whole be raised to the second, third, or fourth,* &c. *power, according to the aforesaid cases, respectively, and subtracted from the given quantity; and, if any thing remain, let the operation be repeated, by always dividing the first term of the remainder by the same divisor, found as above.*

Suppose, for example, it were required to extract the square root of the compound quantity $2ax + a^2 + x^2$: then, having ranged the terms in order, according to the dimensions of the letter a, the given quantity will stand thus, $a^2 + 2ax + x^2$, and the root of its first term will be a; by the double of which I divide $2ax$, the first of the remaining terms, and add $+x$, the quantity thence arising to a, already found, and so have $a + x$ in the quotient; which being raised to the second power, and subtracted from the given quantity, nothing remains; therefore $a+x$ is the square root required. See the operation.

$$\begin{array}{l} a^2+2ax+x^2\ (a+x \\ 2a)\ 2ax \\ \underline{a^2+2ax+x^2}, \text{ second power of } a+x. \\ \quad 0 \quad\ 0 \quad\ 0 \end{array}$$

In like manner, if the quantity $a^4-2a^3x+3a^2x^2-2ax^3+x^4$ be proposed, to extract the square root thereof; the answer will come out a^2-ax+x^2, as appears by the process.

$$\begin{array}{l} a^4-2a^3x+3a^2x^2-2ax^3+x^4\ (a^2-ax+x^2 \\ 2a^2)-2a^3x \\ \underline{a^4-2a^3x+\ a^2x^2}, \text{ second power of } a^2-ax. \\ \quad 2a^2)\quad 2a^2x^2, \text{ first term of the remainder.} \\ \underline{a^4-2a^3x+3a^2x^2-2ax^3+x^4}, \text{ square of } a^2-ax+x^2 \\ 0 \quad\ 0 \quad\ 0 \quad\ 0 \quad\ 0 \end{array}$$

Again, let it be required to extract the cube root of $a^3-6a^2x+12ax^2-8x^3$, and the work will stand thus:

$$\begin{array}{l} a^3\ -6a^2x+12ax^2-8x^3\ (a-2x \\ 3a^2)-6a^2x \\ \underline{a^3\ -6a^2x+12ax^2-8x^3}, \text{ cube of } a-2x. \\ 0 \quad\ 0 \quad\ 0 \quad\ 0 \end{array}$$

Lastly, let it be required to extract the biquadratic root of $16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$, and the process will stand as follows:

$$\begin{array}{l} 16x^4\ -96x^3y+216x^2y^2-216xy^3+81y^4\ (2x-3y \\ 32x^3)-96x^3y \\ \underline{16x^4\ -96x^3y+216x^2y^2-216xy^3+81y^4} \\ 0 \quad\ 0 \quad\ 0 \quad\ 0 \quad\ 0 \end{array}$$

And, in the same manner, the root may be determined in any other case, where it is possible to be extracted; but, if that cannot be done, or if, after all, there be a remainder, then the root is to be expressed in the manner of a surd, according to what has been already shown. As to the truth of the preceding rule, it is too obvious to need a formal demonstration, every operation being a proof of itself. I shall only add here, that there are other rules besides this, for extracting the roots of compound quantities, which sometimes bring out the conclusions rather more expeditiously; but as these are confined to particular cases, and would take up a great deal of room to explain in a manner sufficiently clear and intelligible, it seemed more eligible to lay down the whole in one easy general method, than to discourage and retard the learner by a multiplicity of rules. However, as the extraction of the square root is much more necessary and useful than the rest, I shall here put down one single example thereof, wrought according to the common method of extracting the square root, in numbers: which I suppose the reader to be acquainted with, and which he will find more expeditious than the general rule explained above.

$$\begin{array}{rl} \textit{Examp.} & a^4+4a^3x+6a^2x^2+4ax^3+x^4\,(a^2+2ax+x^2 \\ & a^4 \\ \cline{2-2} 2a^2+2ax) & +4a^3x+6a^2x^2 \\ & +4a^3x+4a^2x^2 \\ \cline{2-2} 2a^2+4ax+x^2) & \qquad\qquad +2a^2x^2+4ax^3+x^4 \\ & \qquad\qquad +2a^2x^2+4ax^3+x^4 \\ \cline{2-2} & \qquad\qquad\quad 0 \qquad\quad 0 \qquad 0 \end{array}$$

SECTION VIII.

Of the Reduction of Fractional and Radical Quantities.

THE reduction of fractional and radical quantities is of use in changing an expression to the most simple and commodious form it is capable of; and that, either by bringing it to its least terms, or all the members thereof, if it be compounded, to the same denomination.

A fraction is reduced to its least terms by dividing both the numerator and denominator by the greatest common divisor.

Thus, $\frac{ab}{bc}$, by dividing by b, is reduced to $\frac{a}{c}$:

And $\frac{2abc}{abb}$, by dividing by ab, is reduced to $\frac{2c}{b}$:

Moreover, $\frac{20abd}{5ab}$ will be reduced to $\frac{4d}{1}$, or $4d$:

And $\frac{12ax^2\sqrt{xy}}{72a^2x^2\sqrt{xy}}$ will be reduced to $\frac{1}{6a}$.

Thus, also, $\frac{12aa - 2ab}{4a^2}$, by dividing every term of the numerator and denominator by $2a$, is reduced to $\frac{6a - b}{2a}$:

And $\frac{8a^3x - 12a^2x^2 + 6ax^3}{6a^2x + 4ax^2}$, by dividing every term by $2ax$, is reduced to $\frac{4a^2 - 6ax + 3x^2}{3a + 2x}$:

Lastly, $\frac{a^3 + 3a^2b + 3ab^2 + b^3}{a^2 + 3ab + 2b^2}$, by dividing both the numerator and denominator by the compound divisor $a + b$, is reduced to $\frac{a^2 + 2ab + bb}{a + 2b}$.

But the compound divisors, whereby a fraction can sometimes be reduced to lower terms, are not so easily discovered as its simple ones; for which reason it may not be improper to lay down a rule for finding such divisors.

First, divide both the numerator and denominator by their greatest simple divisors, and then the quotients one by the other (as is taught in case 7, section 5), *always observing to make that the divisor which is of the least dimensions; and, if any thing remain, divide it by its greatest simple divisor, and then divide the last compound di-*

visor by the quantity thence arising; and if any thing yet remain, divide it likewise by its greatest simple divisor, and the last compound divisor by the quantity thence arising; proceed on in this manner till nothing remains; so shall the last divisor exactly divide both the numerator and denominator, without leaving any remainder.

Note. If, after you have divided any remainder by its simple divisor, you can discover a compound one which will likewise measure the same, and is prime to the divisor from whence that remainder arose, it will be convenient to divide also thereby. And, if in any case it should happen that the first term of the divisor does not exactly measure that of the dividend, the whole dividend may be multiplied by any quantity that shall be necessary to make the operation succeed.

Ex. 1. Let it be required to reduce the fraction $\frac{5a^5+10a^4b+5a^3b^2}{a^3b+2a^2b^2+2ab^3+b^4}$ to its lowest terms, or to find the greatest common measure of its numerator and denominator. Here, dividing first by the greatest simple divisors, $5a^3$ and b, we have $a^2+2ab+b^2$, and $a^3+2a^2b+2ab^2+b^3$: and if the latter of these be divided by the former, the work will stand thus:

$$\begin{array}{l} a^2+2ab+b^2)\ a^3+2a^2b+2ab^2+b^3\ (a \\ \qquad\qquad\qquad\ \ \underline{a^3+2a^2b+\ \ ab^2} \end{array}$$

where the remainder is $\qquad +\ ab^2+b^3$; which being divided by b^2, its greatest simple divisor, gives $a+b$; by this divide $a^2+2ab+b^2$, and the quotient will come out $a+b$, exactly; therefore the last divisor, $a+b$, will exactly measure both quantities, as may be proved thus:

$$\begin{array}{l} a+b)\ 5a^5+10a^4b+5a^3b^2\ (5a^4+5a^3b \\ \qquad\ \ \underline{5a^5+\ \ 5a^4b\qquad\qquad} \\ \qquad\qquad\quad\ 5a^4b+5a^3b^2 \\ \qquad\qquad\quad\ \underline{5a^4b+5a^3b^2} \\ \qquad\qquad\qquad 0\qquad\ 0 \end{array}$$

$$\begin{array}{r}
a+b)\ a^3b+2a^2b^2+2ab^3+b^4\ (a^2b+ab^2+b^3 \\
\underline{a^3b+\ a^2b^2\qquad\qquad\qquad} \\
a^2b^2+2ab^3\qquad \\
\underline{a^2b^2+\ ab^3\qquad} \\
ab^3+b^4 \\
\underline{ab^3+b^4} \\
0\quad 0
\end{array}$$

In both which cases nothing remains; therefore the fraction given will be reduced to $\frac{5a^4+5a^3b}{a^2b+ab^2+b^3}$.

Ex. 2. Let it be proposed to reduce the fraction $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ to its lowest terms: and then the work will stand as follows:

$$\begin{array}{r}
a^3-a^2x-ax^2+x^3)\ a^4+0+0+0-x^4\ (a+x \\
\underline{a^4-a^3x-a^2x^2+ax^3\qquad} \\
a^3x+a^2x^2-ax^3-x^4 \\
\underline{a^3x-a^2x^2-ax^3+x^4} \\
+2a^2x^2+0-2x^4 \\
a^2+0-x^2)\ a^3-a^2x-ax^2+x^3\ (a-x \\
\underline{a^3-0-ax^2\qquad\qquad} \\
-a^2x+0+x^3\qquad \\
\underline{-a^2x+0+x^3\qquad} \\
0\quad 0\quad 0\qquad
\end{array}$$

From whence it appears that a^2+0-x^2, or a^2-x^2, will measure both a^4-x^4 and $a^3-a^2x-ax^2+x^3$; and, by dividing thereby, the fraction proposed is reduced to $\frac{a^2+x^2}{a-x}$.

These operations are founded on this principle, *that whatever quantity measures the whole, and one part of another, must do the like by the remaining part.* For that quantity, whatever it be, which measures both the divisor and dividend, in the first example, must evidently measure $a^3+2a^2b+ab^2$ (being a multiple of

Example 3. In the same manner the fraction $\frac{x^4-3ax^3-8a^2x^2+18a^3x-8a^4}{x^3-ax^2-8a^2x+6a^3}$ will be reduced to $\frac{x^2-5ax+4a^2}{x-3a}$. See the process.

$$\begin{array}{rr}
x^3-ax^2-8a^2x+6a^3) & x^4-3ax^3-8a^2x^2+18a^3x-8a^4(x-2a \\
 & x^4-\ ax^3-8a^2x^2+\ 6a^3x \qquad\qquad \\ \cline{2-2}
 & -2ax^3+\ 0\ +12a^3x-\ 8a^4 \qquad\quad \\
 & -2ax^3+2a^2x^2+16a^3x-12a^4 \qquad\quad \\ \cline{2-2}
\text{remainder} & -2a^2x^2-\ 4a^3x-\ 4a^4; \qquad\quad
\end{array}$$

which divided by $-2a^2$, gives $x^2+2ax-2a^2$ for the next divisor.

$$\begin{array}{rl}
x^2+2ax-2a^2) & x^3-ax^2-8a^2x+6a^3\,(x-3a \\
 & x^3+2ax^2-2a^2x \\ \cline{2-2}
 & -\ 3ax^2-6a^2x+6a^3 \\
 & -\ 3ax^2-6a^2x+6a^3 \\ \cline{2-2}
 & \qquad 0 \qquad 0 \qquad 0
\end{array}$$

$$\begin{array}{rl}
x^2+2ax-2a^2) & x^4-3ax^3-8a^2x^2+18a^3x-8a^4(x^2-5ax+4a^2 \\
 & x^4+2ax^3-2a^2x^2 \\ \cline{2-2}
 & -5ax^3-\ 6a^2x^2+18a^3x \\
 & -5ax^3-10a^2x^2+10a^3x \\ \cline{2-2}
 & \qquad +\ 4a^2x^2+\ 8a^3x-8a^4 \\
 & \qquad +\ 4a^2x^2+\ 8a^3x-8a^4 \\ \cline{2-2}
 & \qquad\quad 0 \qquad\quad 0 \qquad\quad 0
\end{array}$$

Now if, by proceeding in this manner, no compound divisor can be found, that is, if the last remainder be only a simple quantity, we may conclude the case proposed does not admit of any, but is already in its lowest

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the former): whence, by the principle above quoted, the same quantity, as it measures the whole dividend, must also measure the remaining part of it, ab^2+b^3: but, the divisor we are in quest of being a compound one, we may cast off the simple divisor b^2, as not for our purpose: whence $a+b$ appears to be the only compound divisor the case admits of: which, therefore, must be the common measure required, if the example proposed admits of any such.

terms. Thus, for instance, if the fraction proposed were to be $\frac{a^3 + 2a^2x + 3ax^2 + 4x^3}{a^2 + ax + x^2}$; it is plain by inspection, that it is not reducible by any simple divisor; but to know whether it may not, by a compound one, I proceed as above, and find the last remainder to be the simple quantity $7xx$: whence I conclude that the fraction is already in its lowest terms.

Another observation may be here made, in relation to fractions that have in them more than two different letters. When one of the letters rises only to a single dimension, either in the numerator or in the denominator, it will be best to divide the said numerator or denominator (whichever it is) into two parts, so that the said letter may be found in every term of the one part, and be totally excluded out of the other; this being done, let the greatest common divisor of these two parts be found; which will, evidently, be a divisor to the whole, and by which the division of the other quantity is to be tried; as in the following example, where the fraction given is $\frac{x^3 + ax^2 + bx^2 - 2a^2x + bax - 2ba^2}{xx - bx + 2ax - 2ab}$. Here the denominator being the least compounded, and b rising therein to a single dimension only, I divide the same into the parts $x^2 + 2ax$, and $-bx - 2ab$; which, by inspection, appear to be equal to $\overline{x + 2a} \times x$, and $\overline{x + 2a} \times -b$. Therefore $x + 2a$ is a divisor to both the parts, and likewise to the whole, expressed by $\overline{x + 2a} \times \overline{x - b}$; so that one of these two factors, if the fraction given can be reduced to lower terms, must also measure the numerator: but the former will be found to succeed, the quotient coming out $x^2 - ax + bx - ab$, exactly: whence the fraction itself is reduced to $\frac{x^2 - ax + bx - ab}{x - b}$; which is not reducible farther, by $x - b$, since the division does not terminate without a remainder, as upon trial will be found.

Having insisted largely on the reduction of fractions to their least terms, we now come to consider their reduction to the same denominator.

Fractions are reduced to the same denominator by multiplying the numerator of each into all the denominators, except its own, for a new corresponding numerator, and all the denominators continually together, for a common denominator.

Thus, $\frac{a}{b}$ and $\frac{c}{d}$ will be reduced to $\frac{ad}{bd}$ and $\frac{bc}{bd}$;

$\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$, to $\frac{adf}{bdf}$, $\frac{cbf}{bdf}$, and $\frac{bde}{bdf}$;

and $\frac{2ax}{cd}$, and $\frac{5bx}{3a}$, to $\frac{6a^2x}{3acd}$, and $\frac{5bxcd}{3acd}$; and so of others.

But when the denominators have a common divisor, the operation will be more simple, and the conclusion neater, if, instead of multiplying the terms of each fraction by the denominator of the other, you only multiply by that part which arises by dividing by the common divisor. As, if there were proposed the fractions $\frac{b^2}{ad}$ and $\frac{ab}{cd}$; then, the denominators having the factor d common to both, I multiply by the remaining factors a and c; whence the two fractions will be reduced to $\frac{bbc}{acd}$ and $\frac{aab}{acd}$ (where d remains as before, nothing having been done therewith). By the same method $\frac{3xy^3}{5abc}$ and $\frac{7bx^3}{4abd}$ are reduced to $\frac{12dxy^3}{20abcd}$ and $\frac{35bcx^3}{20abcd}$; and $\frac{6a\sqrt{ax}}{5bc}$ and $\frac{7c\sqrt{aa+xx}}{3ab}$, to $\frac{18a^2\sqrt{ax}}{15abc}$ and $\frac{35c^2\sqrt{aa+xx}}{15abc}$.

But, as has been before hinted, the principal use of this sort of reduction is to transform compound quantities to the most commodious forms of expression;

which, for the general part, are more easily managed (whether they are to be added, subtracted, multiplied, or divided) when all their members are brought to the same denomination.

Thus the compound quantity $\frac{a}{b}+\frac{c}{d}$ will be transformed to $\frac{da}{bd}+\frac{bc}{bd}$ or to $\frac{ad+bc}{bd}$; for it is evident, that the quotient which arises by dividing the whole, is equal to the quotients of all the parts, by the same divisor.

In the same manner will $\frac{a}{b}-\frac{c}{d}$ be $=\frac{ad-bc}{bd}$;
and $\frac{2xy}{5aa}+\frac{3x}{b}-\frac{c}{d}=\frac{2xybd+15aaxd-5aabc}{5aabd}$;
also, $\frac{2dx}{a}+b$, or $\frac{2dx}{a}+\frac{b}{1}$ will be $=\frac{2dx+ab}{a}$;
and $\frac{2ab}{a-b}+a=\frac{2ab+aa-ab}{a-b}=\frac{ab+aa}{a-b}$:

So likewise, by reduction, $\frac{a^2}{a-x}+\frac{a^2}{a+x}-2a$ will be $=\frac{a^2\times\overline{a+x}+a^2\times\overline{a-x}-2a\times\overline{a+x}\times\overline{a-x}}{\overline{a-x}\times\overline{a+x}}$ $=\frac{2ax^2}{a^2-x^2}$; and $\frac{5\sqrt{xy}}{a}+\frac{10a-5x}{\sqrt{xy}+a}=$ $\frac{5xy+5a\sqrt{xy}+10a^2-5ax}{a\sqrt{xy}+a^2}$; and so in other cases.

The reason of the two kinds of reduction hitherto explained, is grounded on this obvious principle, that the equimultiples, or like parts of quantities, are in the same ratio to each other, as the quantities themselves; or, that the quotient which arises by dividing one quantity by another, is the same that arises by dividing any part or multiple of the former, by the like part or multiple of the

Besides these, there are yet two other sorts of reduction which authors have treated of under the head of fractions; which are, *the reducing of a whole quantity to an equivalent fraction of a given denomination*, and *a compound fraction to a simple one of the same value*. Neither of these, indeed, are of any great use in the solution of problems; however, it might be improper to leave them entirely untouched.

1°. *A whole quantity is reduced to an equivalent fraction by multiplying it by the given denominator, and writing the multiplier underneath the product, with a line between them.*

Thus the quantity a, reduced to the denominator b, will be $\frac{ab}{b}$, and the quantity $c+d$, to the denominator $a+b$, will be $\frac{\overline{a+b}\times\overline{c+d}}{a+b}$, or $\frac{ac+bc+ad+bd}{a+b}$.

2°. *A compound fraction is reduced to a simple one of the same value by multiplying the numerators together for a new numerator, and the denominators together for a new denominator.*

But by a compound fraction here, we are not to un-

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latter: for, in reducing to the lowest terms, it is plain, that, instead of the whole numerator and denominator, we only take that part of each which is defined by the greatest common measure; whereas, in reduction to the same denominator, we, on the contrary, make use of equimultiples of those quantities; since, in multiplying any numerator into all the denominators, except its own, we multiply it by the very same quantities by which its denominator is multiplied.

The rule for reducing a compound fraction to a simple one, may be explained thus. It is plain that the part of $\frac{c}{d}$ defined by $\frac{1}{b}$, which arises by dividing by b, will be equal to $\frac{c}{bd}$ (the divisor here derstand one consisting of several terms, connected together by the signs + and — (which is the general definition of a compound quantity), but such a one as expresses a given part of some other fraction.

Thus $\frac{2}{3}$ of $\frac{3a}{5}$ will be equal to $\frac{6a}{15}$, and the $\frac{a}{b}$ part of $\frac{c}{d}$ will be $= \frac{ac}{bd}$.

Of the Reduction of Radical Quantities.

The reduction of surd quantities, like that of fractions, may be either to the least terms, or to the same denomination.

A radical quantity is reduced to its least terms by resolving it into factors, and extracting the root of that which is rational.

Thus, $\sqrt{28}$ is reduced to $\sqrt{4} \times \sqrt{7}$; which, by extracting the square root of 4, becomes $2\sqrt{7}$; also $\sqrt{a^2b}$ is reduced to $\sqrt{a^2} \times \sqrt{b}$; which, by extracting the root of a^2, becomes $a\sqrt{b}$: likewise, $\sqrt[3]{a^3b^4c^2}$, or $\overline{a^3b^4c^2}\,]^{\frac{1}{3}}$ is reduced to $\sqrt[3]{a^3b^3} \times \sqrt[3]{bc^2}$, or $ab\sqrt[3]{bc^2}$: moreover, $\sqrt{\frac{4a^3x - 4a^2x^2}{bc^2}}$ is reduced to $\sqrt{\frac{4a^2}{c^2}} \times \sqrt{\frac{ax - x^2}{b}}$, or $\frac{2a}{c} \times \sqrt{\frac{ax - x^2}{b}}$; and $\sqrt[4]{\frac{16a^5x^7 + 16a^6x^6}{81b^4c - 162b^5x}}$

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being b times as great); therefore the part of $\frac{c}{d}$ defined by $\frac{a}{b}$, being a times as great as that defined by $\frac{1}{b}$, must be truly expressed by $\frac{c}{bd} \times a$, or its equal $\frac{ac}{bd}$; *as was to be shown.*

is reduced to $\sqrt[4]{\frac{16a^4x^4}{81b^4}} \times \sqrt[4]{\frac{ax^3 + a^2x^3}{c^2 - 2bx}}$, or $\frac{2ax}{3b} \times \sqrt[4]{\frac{az^3 + a^3x^3}{c^2 - 2bx}}$; and so of any other: all which is evident from case 4 of multiplication, and case 3 of involution. But it is to be observed, that, in resolving any expression in this manner, the factor out of which the root is to be extracted, is always to be taken the greatest the case will admit of. It also may be proper to take notice, that this kind of reduction is chiefly useful in the addition and subtraction of surd quantities, and in uniting the terms of compound expressions that are commensurable to each other, where the irrational part, or factor, after reduction, is the same in each term.

Thus, $\sqrt{18} + \sqrt{32}$ is reduced to $3\sqrt{2} + 4\sqrt{2}$, or $7\sqrt{2}$; and $\sqrt{8a^2} + \sqrt{50a^2} - \sqrt{72a^2}$ is reduced to $2a\sqrt{2} + 5a\sqrt{2} - 6a\sqrt{2} = a\sqrt{2}$. Moreover, by reduction, $\sqrt{\frac{12a^2x}{5}} + \sqrt{\frac{15a^2x}{4}}$ becomes $= \sqrt{\frac{48a^2x}{20}} + \sqrt{\frac{75a^2x}{20}} = 4a\sqrt{\frac{3x}{20}} + 5a\sqrt{\frac{3x}{20}} = 9a\sqrt{\frac{3x}{20}}$.

And $3a\sqrt{4a^2x^2 + 8x^4} + 3x\sqrt{9a + 18a^2x^2}$ becomes $6ax\sqrt{a^2 + 2x^2} + 9ax\sqrt{a^2 + 2x^2} = 15ax\sqrt{a^2 + 2x^2}$.

Surd quantities, under different radical signs, are reduced to the same radical sign, by reducing their indices to the least common denominator.

Thus $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$, reduced to the same radical sign, will become $a^{\frac{3}{6}}$ and $a^{\frac{2}{6}}$ (for the indices are here $\frac{1}{2}$ and $\frac{1}{3}$, and these are equivalent to $\frac{3}{6}$ and $\frac{2}{6}$, where both have the same denominator). In the same manner, $\overline{2}|^{\frac{1}{2}}$ and $\overline{3}|^{\frac{1}{3}}$ will become $2^{\frac{3}{6}}$ and $\overline{3}|^{\frac{2}{6}}$, or $\overline{8}|^{\frac{1}{6}}$ and $\overline{9}|^{\frac{1}{6}}$. And, *universally*, $A^{\frac{n}{m}}$ and $B^{\frac{p}{q}}$ will, when their exponents are reduced to the same denomination, become $\overline{A}\rceil^{\frac{nq}{mq}}$ and $\overline{B}\rceil^{\frac{mp}{mq}}$.

The principal use of this sort of reduction is, when quantities under different radical signs are to be multiplied or divided by each other.

That the reduction of a radical quantity to another of a different denomination, by an equal multiplication of the terms of its exponent, makes no alteration in the value of the quantity, may be thus demonstrated.

Let $A^{\frac{m}{n}}$ be any quantity of this kind; then, the terms of its exponent being equally multiplied by any number r, I say, the quantity $A^{\frac{mr}{nr}}$, hence arising, is equal to the given one $A^{\frac{m}{n}}$.

For, if x be assumed $= A^{\frac{1}{nr}}$, or, which is the same thing, if the value of x be such, that $x^{nr} = A$; then the nth root of x^{nr} being x^{r} (*by case* 2 *of section* 6) and the nth root of A being $A^{\frac{1}{n}}$ (*by notation*), these two quantities x^{r} and $A^{\frac{1}{n}}$ must, likewise, be equal to each other: and, if they be both raised to the nth power, the equality will still continue; but the nth power of the former (x^{r}) is $= x^{mr}$ (*by case* 2 *of involution*); and the mth power of the latter ($A^{\frac{1}{n}}$) is $A^{\frac{m}{n}}$ (*by notation*); therefore x^{mr} is $= A^{\frac{m}{n}}$. But, x being $= A^{\frac{1}{nr}}$, we have $x^{mr} = A^{\frac{mr}{nr}}$, *by notation;* and consequently $A^{\frac{mr}{nr}} = A^{\frac{m}{n}}$; *which was to be proved.*

Thus, $\sqrt{5}$, multiplied by $\sqrt[3]{10}$, or $\overline{5}|^{\frac{1}{2}}$, into $\overline{10}|^{\frac{1}{3}}$, will give $\overline{125}|^{\frac{1}{6}} \times \overline{100}|^{\frac{1}{6}}$, or $\overline{12500}|^{\frac{1}{6}}$: also, $\sqrt{ax}$ into $\sqrt[3]{a^2x}$, or $\overline{ax}|^{\frac{1}{2}}$ into $\overline{a^2x}|^{\frac{1}{3}}$, will give $\overline{a^3x^3}|^{\frac{1}{6}} \times \overline{a^4x^2}|^{\frac{1}{6}}$, or $\overline{a^7x^5}|^{\frac{1}{6}}$: and $\sqrt{ax}$, divided by $\sqrt[3]{a^2x}$ will give $\frac{\overline{a^3x^3}|^{\frac{1}{6}}}{\overline{a^4x^2}|^{\frac{1}{6}}}$, or $\overline{\frac{x}{a}}|^{\frac{1}{6}}$. Lastly, $2x$, multiplied into $\sqrt{3ax}$, will give $\sqrt{4x^2} \times \sqrt{3ax}$, or $\sqrt{12ax^3}$.

SECTION IX.

Of Equations.

AN equation is, when two equal quantities, differently expressed, are compared together, by means of the sign $=$ placed between them.

Thus, $8 - 2 = 6$ is an equation, expressing the equality of the quantities $8 - 2$, and 6: and $x = a + b$ is an equation, showing that the quantity represented by x is equal to the sum of the two quantities represented by a and b.

Equations are the means whereby we come at such conclusions as answer the conditions of a problem; wherein, from the quantities given, the unknown ones are determined; and this is called the resolution or reduction of equations.

Reduction of Single Equations.

Single equations are such as contain only one unknown quantity; which, before that quantity can be discovered, must be so ordered and transformed, by the addition, subtraction, multiplication, or division, *&c.* of equal quantities, that a just equality between the two parts thereof may be still preserved, and that there may result, at last, an equation, wherein the unknown quantity stands alone on one side, and all the known ones on the other. But, though this method of ordering an equation is grounded upon self-evident principles, yet the operations are sometimes a little difficult to manage in the best manner; for which reason the following rules are subjoined.

1°. *Any term of an equation may be transposed to the contrary side, if its sign be changed**.

Thus, if $x + 6 = 16$; then will $x = 16 - 6$, that is, $x = 10$:
And, if $x - 4 = 8$; then will $x = 8 + 4$, or $x = 12$:
Also, if $3x = 2x + 24$; then will $3x - 2x = 24$, that is, $x = 24$:
Again, if $5x - 8 = 3x + 20$; then will $5x - 3x = 20 + 8$, or $2x = 28$:
Lastly, if $ax + bx - c + d - ex = f - g + hx - kx$; then, by transposition, $ax + bx - ex - hx + kx = f - g + c - d$; where all the terms affected by x (the unknown quantity) stand now on the same side of the equation.

2°. *If there be any quantity by which all the terms of the equation are multiplied, let them all be divided by that quantity; but if all of them be divided by any quantity, let the common divisor be cast away.*

Thus, the equation $ax = ab$ is reduced to $x = b$; also, $10x = 70$ (or $10 \times x = 10 \times 7$) is reduced to $x = 7$; and $x^3 = ax^2 + bx^3$ is reduced $x = a + b$:

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* The reason of this rule is extremely evident; since transposing a quantity thus is nothing more than subtracting or adding it on both sides of the equation, according as the sign thereof is positive or negative. Thus, in the equation $x + 6 = 16$ (which, by transposition, becomes $x = 16 - 6 = 10$), the number 6 is subtracted from both sides; and in the equation $x - 4 = 8$ (which, by transposition, becomes $x = 8 + 4 = 12$), the number 4 is added on both sides.

Moreover (*by the latter part of the rule*), $\frac{x}{a} = \frac{b}{a}$ is reduced to $x = b$; and $\frac{ax^3}{c} = \frac{abx^2 - acx^2}{c}$, to $ax^3 = abx^2 - acx^2$; which, if the whole be divided by ax^2, will be farther reduced to $x = b - c$.

3°. *If there be irreducible fractions, let the whole equation be multiplied by the product of all their denominators, or, which is the same thing, let the numerator of every term in the equation be multiplied by all the denominators, except its own, supposing such terms (if any there be) that stand without a denominator, to have a unit subscribed.*

Thus, the equation $x + \frac{x}{2} + \frac{x}{3} = 11$, is reduced to $6x + 3x + 2x = 66$; and $x + \frac{x+2}{5} = 12 + \frac{x-3}{8}$, to $40x + 8x + 16 = 480 + 5x - 15$: so, likewise, $a - \frac{x}{a} = \frac{x+b}{c}$ is reduced to $a^2c - cx = ax + ab$; and $\frac{ax}{a+x} + a = \frac{cx}{b}$ to $abx + a^2b + abx = acx + cx^2$.

4°. *If, in your equation, there be an irreducible surd, wherein the unknown quantity enters, let all the other terms be transposed to the contrary side* (by rule 1); *and then, if both sides be involved to the power denominated by the surd, an equation will arise free from radical quantities; unless there happen to be more surds than one, in which case the operation is to be repeated.*

Thus, $\sqrt{x} + 6 = 10$, by transposition, becomes $\sqrt{x}$ $(= 10 - 6) = 4$; which, by squaring both sides, gives $x = 16$.

So, likewise, $\sqrt{aa + xx} - c = x$, becomes $\sqrt{aa + xx} = c + x$; which, squared, gives $aa + xx = cc + 2cx + xx$, or $aa - cc = 2cx$ (*by rule* 1). The reasons of this, as well as of the two preceding rules, depend on self-evident principles: for, when the equal quantities, on each side of an equation, are multiplied or divided by the same, or by equal quantities, or raised to equal powers, the quantities resulting must necessarily be equal.

5°. *Having, by the preceding rules (if there be occasion), cleared your equation of fractional and radical quantities, and so ordered it, by transposition, that all the terms, wherein the unknown quantity is found, may stand on the same side thereof, let the whole be divided by the coefficient, or the sum of the coefficients, of the highest power of the said unknown quantity:* and then, if your equation be a simple one (that is, if the first power, or the quantity itself, be only concerned), the work is at an end; but if it be a quadratic or cubic one, &c. something further remains to be done; and recourse must be had to the particular methods for resolving these kinds of equations, hereafter to be considered in their proper place.

I shall here subjoin a few examples for the learner's exercise, wherein all the foregoing rules occur promiscuously.

Ex. 1. Let $5x - 16 = 3x + 12$: then (*by rule* 1) $5x - 3x = 12 + 16$, or $2x = 28$: whence (*by rule* 5) $x = \frac{28}{2} = 14$.

Ex. 2. Let $20 - 3x - 8 = 60 - 7x$: then, $-3x + 7x = 60 - 20 + 8$, that is, $4x = 48$; and consequently $x = \frac{48}{4} = 12$.

Ex. 3. Let $ax - b = cx + d$; then $ax - cx = d + b$, or $\overline{a - c} \times x = d + b$; and therefore $x = \frac{d + b}{a - c}$ (*by rule* 5).

Ex. 4. Let $6x^2 - 20x = 16x + 2x^2$: then, dividing by $2x$ (*according to rule* 2) we have $3x - 10 = 8 + x$; whence $3x - x = 8 + 10$, that is, $2x = 18$; and therefore $x = \frac{18}{2} = 9$.

Ex. 5. Let $3ax^3 - abx^2 = ax^3 + 2acx^2$: here, dividing the whole by ax^2, we have $3x - b = x + 2c$; therefore $2x = 2c + b$, and $x = \frac{2c+b}{2}$.

Ex. 6. Let $\frac{x}{3} + \frac{x}{4} = 21$: then (*by rule* 3) $4x + 3x = 252$; and therefore $x = \frac{252}{7} = 36$.

Ex. 7. Let $\frac{x+1}{2} + \frac{x+2}{3} = 16 - \frac{x+3}{4}$: then $12x + 12 + 8x + 16 = 384 - 6x - 18$: whence $26x = 338$, and $x = \frac{338}{26} = 13$.

Ex. 8. Let $a - \frac{bb}{x} = c$: then $ax - bb = cx$; whence $ax - cx = bb$, and $x = \frac{bb}{a-c}$.

Ex. 9. Let $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = d$: then, $bcx + acx + abx = abcd$, or $\overline{bc+ac+ab} \times x = abcd$; and consequently $x = \frac{abcd}{bc+ac+ab}$ (*by rule* 5).

Ex. 10. Let $ax + b^2 = \frac{ax^2+ac^2}{a+x}$: then, $\overline{ax+b^2} \times \overline{a+x} = ax^2 + ac^2$, that is, $a^2x + ab^2 + ax^2 + b^2x = ax^2 + ac^2$; whence $a^2x + ax^2 + b^2x - ax^2 = ac^2 - ab^2$, or $a^2x + b^2x = ac^2 - ab^2$; and therefore $x = \frac{ac^2 - ab^2}{aa + bb}$.

Ex. 11. Let $\frac{a}{a+x} + \frac{b}{x} = 1$: then $ax + ab + bx = ax + xx$; whence $-xx + bx = -ab$: which, by changing all the signs (in order that the highest power of x may be positive), gives $xx - bx = ab$. But the same conclusion may be otherwise brought out, by first changing the sides of the equation $ax + ab + bx = ax$

$+ xx$; which thereby becoming $ax + xx = ax + ab + bx$, we thence get $xx - bx = ab$, *the same as before.*

Ex. 12. Let $\frac{\sqrt{5x}}{3} + 12 = 17$: then $\frac{\sqrt{5x}}{3} = 5$, and $\sqrt{5x} = 15$; whence (*by rule* 4), $5x = 225$, and therefore $x = \frac{225}{5} = 45$.

Ex. 13. Let $\sqrt{12 + x} = 2 + \sqrt{x}$: then (*by rule* 4) $12 + x = 4 + 4\sqrt{x} + x$; whence, by transposition, $8 = 4\sqrt{x}$; and, by division, $2 = \sqrt{x}$; consequently, $4 = x$.

Ex. 14. Let $x + \sqrt{a^2 + x^2} = \frac{2a^2}{\sqrt{a^2 + x^2}}$. Here (*by rule* 3) $x \times \sqrt{a^2 + x^2} \times a^2 + x^2 = 2a^2$; whence $x \times \sqrt{a^2 + x^2} = a^2 - x^2$, and $x^2 \times \overline{a^2 + x^2} = a^4 - 2a^2x^2 + x^4$ (*by rule* 4), that is, $a^2x^2 + x^4 = a^4 - a^2x^2 + x^4$; therefore $3a^2x^2 = a^4$, and $x^2 = \frac{a^4}{3a^2} = \frac{a^2}{3}$.

Ex. 15. Let $\sqrt{x} + \sqrt{a + x} = \frac{2a}{\sqrt{a + x}}$. Then $\sqrt{ax + xx} + a + x = 2a$, or $\sqrt{ax + xx} = a - x$; whence $ax + xx = a^2 - 2ax + x^2$, and $x = \frac{a^2}{3a} = \frac{a}{3}$.

Ex. 16. Let $\sqrt[3]{x^3 - a^3} = x - c$: then, by cubing both sides, $x^3 - a^3 = x^3 - 3cx^2 + 3c^2x - c^3$; whence $3cx^2 - 3c^2x = a^3 - c^3$, and $x^2 - cx = \frac{a^3}{3c} - \frac{c^2}{3}$, by dividing the whole by $3c$.

Ex. 17. Let $\sqrt{aa + xx} = \sqrt[4]{b^4 + x^4}$; then, by raising both sides to the fourth power we have $\overline{aa + xx}|^2 = b^4 + x^4$, that is, $a^4 + 2a^2x^2 + x^4 = b^4 + x^4$; and consequently $x^2 = \frac{b^4 - a^4}{2a^2} = \frac{b^4}{2aa} - \frac{1}{2}a^2$.

Ex. 18. Let $x = \sqrt{a^2 + x\sqrt{bb + xx}} - a$. Here $x + a = \sqrt{a^2 + x\sqrt{bb + xx}}$; which, squared, gives $x^2 + 2ax + a^2 = a^2 + x\sqrt{bb + xx}$, or $x^2 + 2ax = x\sqrt{bb + xx}$; divide by x, so shall $x + 2a = \sqrt{bb + xx}$; this squared again gives $x^2 + 4ax + 4a^2 = bb + xx$; whence $4ax = bb - 4a^2$, and therefore $x = \frac{bb}{4a} - a$.

Of the Extermination of Unknown Quantities, or the reduction of two or more equations to a single one.

It has been shown above, how to manage a single equation; but it often happens, that, in the solution of the same problem, two, three, or more equations are concerned, and as many unknown quantities occur promiscuously in each of them; which equations, before any one of those quantities can be known, must be reduced into one, or so ordered and connected, that, from thence, a new equation may at length arise, affected with only one unknown quantity. This, in most cases, may be performed various ways, but the following are the most general.

1°. *Observe which of all your unknown quantities is the least involved, and let the value of that quantity be found in each equation (by the methods already explained), looking upon all the rest as known; let the values thus found be put equal to each other (for they are equal, because they all express the same thing); whence new equations will arise, out of which that quantity will be totally excluded; with these new equations the operation may be repeated, and the unknown quantities exterminated, one by one, till, at last, you come to an equation containing only one unknown quantity.*

2°. *Or, let the value of the unknown quantity, which you would first exterminate, be found in that equation wherein it is the least involved, considering all the other quantities as known; and let this value, and its powers, be substituted for that quantity and its respective powers in the* *other equations; and, with the new equations thus arising, repeat the operation, till you have only one unknown quantity and one equation.*

3°. *Or, lastly, let the given equations be multiplied or divided by such numbers or quantities, whether known or unknown, that the term which involves the highest power of the unknown quantity to be exterminated, may be the same in each equation; and then by adding, or subtracting the equations, as occasion shall require, that term will vanish, and a new equation emerge, wherein the number of dimensions (if not the number of unknown quantities) will be diminished.*

But the use of the different methods here laid down will be more clearly understood by help of a few examples.

EXAMPLE I.

Let there be given the equations $x + y = 12$, *and* $5x + 3y = 50$; *to find x and y.*

According to the first method, by transposing y and $3y$, we get $x = 12 - y$, and $5x = 50 - 3y$; from the last of which equations, $x = \frac{50 - 3y}{5}$. Now, by equating these two values of x, we have $12 - y = \frac{50 - 3y}{5}$; and therefore $60 - 5y = 50 - 3y$: from which y is given $= \frac{10}{2} = 5$; and $x\ (= 12 - y = 12 - 5) = 7$.

According to the second method, x being, by the first equation, $= 12 - y$, this value must therefore be substituted in the second, that is, $60 - 5y$ must be written in the room of its equal $5x$; whence will be had $60 - 5y + 3y = 50$: and from thence $y = \frac{10}{2} = 5$, *as before.*

But, according to the third method, having multiplied the first equation by 5, it will stand thus, $5x + 5y = 60$; from whence subtracting the 2d equation, $5x + 3y = 50$, there remains - - $2y = 10$; whence $y = 5$, *still the same as before.*

The first of these three ways is much used by some authors; but the last of them is, for the general part, the most easy and expeditious in practice, and is, for that reason, chiefly regarded in the subsequent examples.

EXAMPLE II.

Let $\begin{cases} 5x + 8y = 124 \\ 3x - 2y = 20. \end{cases}$

Here the second equation being multiplied by 4, in order that the coefficients of y in both equations may be the same, we have $12x - 8y = 80$.

Let this equation and the first be now added together; whence y will be exterminated, there coming out $17x = 204$; from which $x = \frac{204}{17} = 12$: therefore, by the first equation, $y\ (= \frac{124 - 5x}{8} = \frac{124 - 60}{8} = \frac{64}{8}) = 8$.

EXAMPLE III.

Given. $\begin{cases} 5x - 3y = 90 \\ 2x + 5y = 160. \end{cases}$

Here multiplying the first equation by 2, and the second by 5, in order that the coefficient of x may be the same in both, there arises

$$\begin{aligned} 10x - 6y &= 180 \\ 10x + 25y &= 800. \end{aligned}$$

By subtracting the former of which from the latter we have $31y = 620$: hence $y = \frac{620}{31} = 20$; and so, by the first equation, $x\ (= \frac{90 + 3y}{5} = \frac{90 + 60}{5}) = 30$.

But the value of x may be otherwise found, independent of the value of y; for, by multiplying the first equation by 5, and the second by 3, and then adding them together, y will be exterminated, and you will get $25x + 6x = 450 + 480$; whence $x = \frac{930}{31} = 30$, *the same as before.*

EXAMPLE IV.

$$\text{Given.}\left\{\begin{array}{l}\dfrac{x}{2}+\dfrac{y}{3}=16\\[1ex] \dfrac{x}{5}-\dfrac{y}{9}=2\end{array}\right.$$

Here our equations, cleared of fractions, will be

$$3x+2y=96$$
$$9x-5y=90.$$

And, if from the triple of the former the latter be subtracted, we shall have $6y+5y=288-90$, that is, $11y=198$; whence $y=18$; and $x\ (=\dfrac{96-2y}{3})=20$.

EXAMPLE V.

$$\text{Given.}\left\{\begin{array}{l}\dfrac{x}{2}-12=\dfrac{y}{4}+8\\[1ex] \dfrac{x+y}{5}+\dfrac{x}{3}-8=\dfrac{2y-x}{4}+27.\end{array}\right.$$

Here $4x-96=2y+64$, and
$12x+12y+20x-480=30y-15x+1620$; which, contracted, become
$4x-2y=160$, and
$47x-18y=2100$: from the last of which subtract 9 times the former; so shall $11x=2100-1440=660$; therefore $x=60$, and $y\ (=\dfrac{4x-160}{2}=2x-80)=40$.

EXAMPLE VI.

$$\text{Let}\left\{\begin{array}{l}x+y=13\\ x+z=14\\ y+z=15\end{array}\right\}\text{; to find } x,\ y, \text{ and } z.$$

By subtracting the first equation from the second, in order to exterminate x, we have $z-y=1$; to which the third equation being added, y will likewise be exterminated, there coming out $2z=16$, or $z=8$; whence $y\ (=z-1)=7$; and $x\ (=13-y)=6$.

EXAMPLE VII.

$$\text{Let}\left\{\begin{array}{l}\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62\\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47\\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38.\end{array}\right.$$

Here the given equations, cleared of fractions, become

$$\begin{array}{l}12x+8y+6z=1488\\ 20x+15y+12z=2820\\ 30x+24y+20z=4560.\end{array}$$

Now, to exterminate z, let the second of these equations be subtracted from the double of the first; and also the triple of the third from the quintuple of the second; whence is had

$$\begin{array}{l}4x+y=156\\ 10x+3y=420:\end{array}$$

from which $12x - 10x = 468 - 420$, and $x = \frac{48}{2} = 24$.

Therefore $y\ (= 156 - 4x) = 60$; and $z\ (= \frac{1488 - 8y - 12x}{6}) = 120$.

EXAMPLE VIII.

$$\text{Let}\left\{\begin{array}{l}x+100=y+z\\ y+100=2x+2z\\ z+100=3x+3y.\end{array}\right.$$

To the double of the first let the second equation be added; so shall the *x*es, on the contrary sides, destroy each other, and you will have $300 + y = 2y + 4z$, or $300 = y + 4z$. Moreover, to the triple of the first let the third equation be added, whence will be had $z + 400 = 6y + 3z$, or $400 = 6y + 2z$.

Now, if from the double of this last equation the former, $300 = y + 4z$, be subtracted, there will come out $500 = 11y$; and, consequently, $y = \frac{500}{11} = 45\tfrac{5}{11}$;

therefore z $\left(= \frac{300 - y}{4} = 75 - \frac{y}{4} = 75 - 11\frac{4}{11}\right) = 63\frac{7}{11}$; and x $(= y + z - 100 = 109\frac{1}{11} - 100) = 9\frac{1}{11}$.

EXAMPLE IX.

Let $x - y = 2$, and $xy + 5x - 6y = 120$; *to exterminate x.*

By the former equation, $x = y + 2$; which value being substituted in the latter (*according to the second general method*) it becomes $\overline{y+2} \times y + 5 \times \overline{y+2} - 6y = 120$, that is, $y^2 + 2y + 5y + 10 - 6y = 120$, or $y^2 + y = 110$.

EXAMPLE X.

Let there be given $x + y = a$, and $x^2 + y^2 = b$; to exterminate x.

Here, by the first equation, $x = a - y$; and therefore $x^2 = \overline{a-y}|^2$; which value being written in the other equation, we have $\overline{a-y}|^2 + y^2 = b$, that is, $a^2 - 2ay + y^2 + y^2 = b$; and therefore $y^2 - ay = \frac{b - aa}{2}$.

EXAMPLE XI.

Given $\left\{\begin{matrix} axy + bx + cy = d \\ fxy + gx + hy = k \end{matrix}\right\}$ *to exterminate y.*

Multiply the first equation by f, and the second by a, and subtract the latter product from the former; whence you will have $bfx - agx + cfy - ahy = df - ak$; which, by transposition and division, gives $y = \frac{df - ak + agx - bfx}{cf - ah}$.

Let this value of y be now substituted in the first equation, and there will arise

$\frac{adfx - a^2kx + a^2gx^2 - abfx^2 + cdf - cak + cagx - cbfx}{cf - ah} + bx = d$: which, multiplied by $cf - ah$, and contracted, gives $ag - bf \times x^2 + df - ak + cg - bh \times x = ck - hd$.

EXAMPLE XII.

Supposing $ax^2 + bx + c = 0$, and $fx^2 + gx + h = 0$; *to exterminate* x.

Proceeding here as in the last example, we have $fbx + fc - agx - ah = 0$, and, from thence, $x = \frac{ah - fc}{fb - ag}$.

Whence, by substitution, $a \times \frac{\overline{ah - fc}\rceil^2}{\overline{fb - ag}\rceil^2} + \frac{b \times \overline{ah - fc}}{fb - ag} + c = 0$. This, by uniting the two last terms, and dividing the whole by a, gives $\frac{\overline{ah - fc}\rceil^2}{\overline{fb - ag}\rceil^2} + \frac{bh - cg}{fb - ag} = 0$; consequently, $\overline{ah - fc}\rceil^2 + \overline{fb - ag} \times \overline{bh - cg} = 0$.

After the same manner x may be expunged out of the equations $ax^3 + bx^2 + cx + d = 0$, and $fx^2 + gx + h = 0$, &c. But, to show the use of the above example, suppose there were given the equations $x^2 + yx - y^2 = 0$, and $x^2 + 3xy - 10 = 0$: then, by comparing the terms of these equations with those of the general ones, $ax^2 + bx + c = 0$, and $fx^2 + gx + h = 0$; we have $a = 1$, $b = y$, $c = -y^2$, $f = 1$, $g = 3y$, and $h = -10$; which values being substituted in the equation $\overline{ah - fc}\rceil^2 + \overline{fb - ag} \times \overline{bh - cg} = 0$, it thence becomes $\overline{-10 + yy}\rceil^2 + \overline{y - 3y} \times \overline{-10y + 3y^3} = 0$, that is, $100 - 20y^2 + y^4 + 20y^2 - 6y^4 = 0$; or, $100 = 5y^4$; whence y may be found, and from thence the value of x also.

SECTION X.

Of Proportion.

QUANTITIES of the same kind may be compared together, either with regard to their differences, or according to the part or parts that one is of the other, called their ratio. The comparison of quantities according to their differences, is called *arithmetical;* but, according to their ratios, *geometrical.*

When, of four quantities, 2, 6, 12, 16, the difference of the first and second is equal to the difference of the third and fourth, those quantities are said to be *in arithmetical proportion.* But, when the ratio of the first and second is the same with that of the third and fourth (as in 2, 6, 10, 30), then the quantities are said to be *in geometrical proportion.* Moreover, when the difference, or the ratio, of every two adjacent terms (as well of the second and third, as of the first and second, *&c.*) is the same, then the proportion is said to be *continued:* thus, 2, 4, 6, 8, *&c.* is a continued arithmetical proportion; and 2, 4, 8, 16, *&c.* a continued geometrical one. These kinds of proportions are also called progressions, being carried on according to the same law throughout.

Arithmetical Proportion.

THEOREM I.

*Of any four quantities, a, b, c, d, in arithmetical progression**, *the sum of the two means is equal to the sum of the two extremes.*

For since, by supposition, $b - a$ is $= d - c$, therefore is $b + c = d + a$, by transposition.

THEOREM II.

In any continued arithmetical progression (5, 7, 9, 11, 13, 15) *the sum of the two extremes, and that of every other two terms equally distant from them, are equal.*

For since, by the nature of progressionals, the second term exceeds the first by just as much as its corresponding term, the last but one, wants of the last, it is manifest, that, when these corresponding terms are added together, the excess of the one will make good the defect of the other, and so their sum be exactly the same with that of the two extremes: and in the same manner it will appear, that the sum of any other two corresponding terms must be equal to that of the two extremes.

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* Although, in the comparison of quantities according to their differences, the term *proportion* is used; yet the word *progression* is frequently substituted in its room, and is, indeed, more proper; the former term being, in the common acceptation of it, synonymous with *ratio*, which is only used in the other kind of comparison.

When the number of terms is odd, as in the progression, 4, 7, 10, 13, 16, then the sum of the two extremes being double to the middle term, or mean, the sum of any other two terms, equally remote from the extremes, must likewise be double to the mean.

THEOREM III.

In any continued arithmetical progression, a, $a + d$, $a + 2d$, $a + 3d$, $a + 4d$, *&c. the last, or greatest term, is equal to the first, or least, more the common difference of the terms drawn into the number of all the terms after the first, or into the whole number of terms, less one.*

For, since every term, after the first, exceeds that preceding it, by the common difference, it is plain that the last must exceed the first by as many times the common difference as there are terms after the first; and therefore must be equal to the first, and the common difference repeated that number of times.

THEOREM IV.

The sum of any rank or series of quantities, in continued arithmetical progression (5, 7, 9, 11, 13, 15), *is equal to the sum of the two extremes multiplied into half the number of terms.*

For, because, by the second theorem, the sum of the two extremes, and that of every two other terms equally remote from them are equal, the whole series, consisting of half as many such equal sums as there are terms, will therefore be equal to the sum of the two extremes repeated half as many times as there are terms. The same thing also holds good, when the number of terms is odd, as in the series 8, 12, 16, 20, 24; for then the mean, or middle term, being equal to half the sum of any two terms equally distant from it, on contrary sides, it is obvious that the value of the whole series is the same, as if every term thereof was equal to the mean, and therefore is equal to the mean (or half the sum of the two extremes) multiplied by the whole number of terms; or to the whole sum of the extremes multiplied by half the number of terms.

Geometrical Proportion.

THEOREM I.

If four quantities, a, b, c, d (2, 6, 5, 15), be in geometrical proportion, the product of the two means, bc, will be equal to that of the two extremes, ad.

For, since the ratio of a to b (or the part which a is of b) is expressed by $\frac{a}{b}$, and the ratio of c to d, in like manner, by $\frac{c}{d}$; and since, by supposition, these two ratios are equal, let them both be multiplied by bd, and the products $\frac{a}{b} \times bd$ and $\frac{c}{d} \times bd$ will likewise be equal; that is, $\frac{abd}{b} = \frac{cbd}{d}$, or $ad = cb$ (*by case* 2, *sect.* 4).

THEOREM II.

If four quantities, a, b, c, d, be such that the product of two of them, ad, is equal to the product of the other two, bc, then are these quantities proportional.

For since, by supposition, the products ad and bc are equal, let both be divided by bd, and the quotients $\frac{ad}{bd}$ $\left(\frac{a}{b}\right)$ and $\frac{bc}{bd}$ $\left(\frac{c}{d}\right)$ will also be equal; and therefore $a : b :: c : d$.

THEOREM III.

If four quantities, a, b, c, d (2, 6, 5, 15), be proportional, the rectangle of the means, divided by either extreme, will give the other extreme.

For, by the second theorem, $ad = bc$ ($2 \times 15 = 6 \times 5$), whence, dividing both sides of the equation by a (2), we have $d = \frac{bc}{a}$ $\left(15 = \frac{6 \times 5}{2}\right)$. Hence, if the two means and one extreme be given, the other extreme may be found.

THEOREM IV.

The products of the corresponding terms of two geometrical proportions are also proportional.

That is, if $a : b :: c : d$, and $e : f :: g : h$; then will $ae : bf :: cg : dh$.

For $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$, by supposition; whence $\frac{a}{b} \times \frac{e}{f} = \frac{c}{d} \times \frac{g}{h}$, by equal multiplication; and consequently $\frac{ae}{bf} = \frac{cg}{dh}$ (by p. 18); that is, $ae : bf :: cg : dh$. Hence it follows, that, if four quantities be proportional, their squares, cubes, &c. will likewise be proportional.

THEOREM V.

If four quantities, a, b, c, d (2, 6, 5, 15), *be proportional,*

Then,

1. inversely,	$b : a :: d : c$	(6 : 2 :: 15 : 5)
2. alternately,	$a : c :: b : d$	(2 : 5 :: 6 : 15)
3. compoundedly,	$a : a+b :: c : c+d$	(2 : 8 :: 5 : 20)
4. dividedly,	$a : b-a :: c : d-c$	(2 : 4 :: 5 : 10)
5. mixedly,	$b+a : b-a :: d+c : d-c$	(8 : 4 :: 20 : 10)
6. by multiplication,	$ra : rb :: c : d$	($2r : 6r$ :: 5 : 15)
7. by division,	$\frac{a}{r} : \frac{b}{r} :: c : d$	($\frac{2}{r} : \frac{6}{r}$ :: 5 : 15)

Because the product of the means, in each case, is equal to that of the extremes, and therefore the quantities are proportional, by theorem 2.

THEOREM VI.

If three numbers, a, b, c (2, 4, 8), be in continued proportion, the square of the first will be to that of the second as the first number to the third; that is, $a^2 : b^2 :: a : c$.

For, since $a : b :: b : c$, thence will $ac = bb$, by theorem 1; and therefore $aac = abb$, by equal multiplication; consequently $a^2 : b^2 :: a : c$, by theorem 2.

In like manner it may be proved, that of four quantities continually proportional, the cube of the first is to that of the second as the first quantity to the fourth.

THEOREM VII.

In any continued geometrical proportion (1, 3, 9, 27, 81, &c.) the product of the two extremes, and that of every other two terms, equally distant from them, are equal.

For the ratio of the first term to the second, being the same as that of the last but one to the last, these four terms are in proportion; and therefore, by theorem 1, the rectangle of the extremes is equal to that of their two adjacent terms; and, after the very same manner, it will appear, that the rectangle of the third and last but two is equal to that of their two adjacent terms, the second and last but one; and so of the rest. Whence the truth of the proposition is manifest.

THEOREM VIII.

The sum of any number of quantities, in continued geometrical proportion, is equal to the difference of the rectangle of the second and last terms, and the square of the first, divided by the difference of the first and second terms.

For, let the first term of the proportion be denoted by a, the common ratio by r, the number of terms by n, and the sum of the whole progression by x: then it is manifest that the second term will be expressed by $a \times r$, or ar; the third by $ar \times r$, or ar^2; the fourth by $ar^2 \times r$, or ar^3, and the nth or last term by ar^{n-1}; and therefore the proportion will stand thus: $a + ar + ar^2 + ar^3 \dots + ar^{n-2} + ar^{n-1} = x$; which equation, multiplied by r, gives $ar + ar^2 + ar^3 + ar^4 \ldots\ldots + ar^{n-1} + ar^n = rx$; from which the first equation being subtracted there will remain $-a + ar^n = rx - x$; whence $x = \left(\frac{ar^n - a}{r - 1} = \frac{r \times ar^{n-1} - a}{r - 1}\right) = \frac{ar \times ar^{n-1} - aa}{ar - a}$; as was to be demonstrated.

SECTION XI.

The Application of Algebra to the Resolution of Numerical Problems.

WHEN a problem is proposed to be solved algebraically, its true design and signification ought, in the first place, to be perfectly understood, so that, if needful, it may be abstracted from all ambiguous and unnecessary phrases, and the conditions thereof exhibited in the clearest light possible. This being done, and the several quantities therein concerned being denoted by proper symbols, let the true sense and meaning of the question be translated from the verbal to a symbolical form of expression; and the conditions, thus expressed in algebraic terms, will, if it be properly limited, give as many equations as are necessary to its solution. But, if such equations cannot be derived without some previous operations, which frequently happens to be the case, then let the learner observe this rule, *viz.* let him consider what method or process he would use to prove, or satisfy himself in, the truth of the solution, were the numbers that answer the conditions of the question to be given or affirmed to be so and so; and then, by following the very same steps, only using unknown symbols instead of known numbers, the question will be brought to an equation.

Thus, if the question were to find a number, which being multiplied by 5, and 8 subtracted from the product, the square of the remainder shall be 144; then, having put $a = 5$, $b = 8$, and $c = 144$, suppose the number sought

to be - - -	4	(or)	x
then 5, or a times that number will be	20		ax
from which 8, or b, being subtracted, there remains	12		$ax - b$
which squared is - -	144		$a^2x^2 - 2axb + b^2$.

Therefore $a^2x^2 - 2axb + b^2$ is $= c$ (or 144), according to the conditions of the question. In the same manner may a question be brought to an equation when two or more quantities are required.

After the conditions of a problem are noted down in algebraic terms, the next thing to be done is to consider whether it be properly limited, or admits of an indefinite number of answers; in order to discover which, observe the following rules.

RULE I.

When the number of quantities sought exceeds the number of equations given, the question, for the general part, is capable of innumerable answers.

Thus, if it be required to find two numbers (x and y) with this one single condition, that their sum shall be 100, we shall have only one equation, *viz.* $x + y = 100$, but two unknown quantities, x and y, to be determined; therefore it may be concluded that the question will admit of innumerable answers.

RULE II.

But if the number of equations, given from the conditions of the question, be just the same as the number of quantities sought, then is the question truly limited.

As, if the question were to find two numbers, whose sum is 100, and whose difference is 20; then, x being put for the greater number, and y for the less, we shall have $x + y = 100$, and $x - y = 20$: therefore, there being here two equations and two unknown quantities, the question is truly limited; 60 and 40 being the only two numbers that can answer the conditions thereof.

RULE III.

When the number of equations exceeds the number of quantities sought, either the conditions of the problem are inconsistent one with another, or what is proposed in general terms can only be possible in certain particular cases.

But it is to be observed, that the equations understood here, as well as in the preceding rules, are supposed to be no ways dependent upon, or consequences of one another. If this be not the case, the question may be either unlimited or absurd, or perhaps both, at the same time that it seems truly limited; as will appear by the following example:

Wherein it is required to find three numbers, under these conditions, that the sum of once the first, twice the second, and three times the third, may be equal to a given number b; that the sum of four times the first, five times the second, and six times the third, may be equal to a given number c; and that the sum of seven times the first, eight times the second, and nine times the third, may be equal to a third given number d. Now, the three numbers sought being respectively denoted by x, y, and z, the question, in algebraic terms, will stand thus:

$$x + 2y + 3z = b$$
$$4x + 5y + 6z = c$$
$$7x + 8y + 9z = d.$$

Here, there being three equations, and just the same number of unknown quantities, one might conclude the question to be truly limited: but, by reflecting a little upon the nature and form of these equations, the contrary will soon appear; because the last of them includes no new condition but what is comprised in and may be derived from the other two; for if from the double of the second the first equation be taken away, the value of $7x + 8y + 9z$ will from thence be given $= 2c - b$. Hence it is manifest, that giving the value of $7x + 8y + 9z$, in the third equation, contributes nothing towards limiting the problem; and that the problem itself is not only unlimited, but also impossible, except when d is given equal to $2c - b$.

Having laid down the necessary rules for bringing problems to equations, and for discovering when they are truly limited, it remains that we illustrate what is hitherto delivered by proper examples.

Arithmetical Problems.

PROBLEM I.

To find that number, to which 75 being added, the sum shall be the quadruple of the said required number.

Let the number sought be represented by - x;
then will its quadruple be denoted by - $4x$;
whence, by the conditions of the question, $x + 75 = 4x$;
this equation, by transposing x, becomes $75 = 3x$:
from whence, dividing by 3, we have $x = \frac{75}{3} = 25$, which is the number that was to be found; for it is plain that $25 + 25 \times 3 = 25 \times 4 = 100$.

PROBLEM II.

What number is that, which being added to 4, and also multiplied by 4, the product shall be the triple of the sum?

Let the number sought be denoted by - x;
so shall the sum be denoted by - $x + 4$,
and the product by - - - $4x$:
whence, by the conditions of the question, $4x = \overline{x + 4} \times 3$; that is, $4x = 3x + 12$; from which, by transposition, $x = 12$.

PROBLEM III.

To find two numbers such, that their sum shall be 30, and their difference 12.

If x be taken to denote the lesser of the two numbers; then, by adding the difference 12, the greater number will be denoted by $x + 12$; and so we shall have $2x + 12 = 30$, *by the question.*

From which equation, $2x = 30 - 12 = 18$; and consequently, $x = \frac{18}{2} = 9$; whence the greater number $(x + 12)$ is also given $= 21$.

PROBLEM IV.

To divide the number 60 *into three such parts, that the first may exceed the second by* 8, *and the third by* 16.

Let the first part be denoted by x; then the second will be $x - 8$, and the third $x - 16$: the aggregate of all which, or $3x - 24$ is $= 60$, *by the question.*

Hence $3x = 60 + 24 = 84$, and $x = \frac{84}{3} = 28$: so that 28, 20, and 12 are the three parts required.

PROBLEM V.

The sum of 660*l. was raised, for a certain purpose, by four persons,* A, B, C, *and* D: *whereof* B *advanced twice as much as* A; C *as much as* A *and* B; *and* D *as much as* B *and* C: *what did each person contribute?*

Let the sum or number of pounds advanced by A be called x;
then will the number of B's pounds be denoted by $2x$;
that of C's by - - - $3x$;
and that of D's by - - $5x$;
the sum of all which is given equal to 660*l.* that is, $11x = 660$: from whence $x = \frac{660}{11} = 60$. Therefore 60, 120, 180, and 300*l.* are the respective sums that were to be determined.

PROBLEM VI.

A certain sum of money was shared among five persons, A, B, C, D, *and* E; *whereof* B *received* 10*l. less than* A; C 16*l. more than* B; D 5*l. less than* C; E 15*l. more than* D: *moreover it appeared that the shares of the two last together were equal to the sum of the shares of the other three: what was the whole sum shared, and how much did each receive?*

Let x denote the share of A:

$$\text{then}\left\{\begin{array}{l} x-10 \\ x+\ 6 \\ x+\ 1 \\ x+16 \end{array}\right\}\text{will be the share of}\left\{\begin{array}{l} \text{B}, \\ \text{C}, \\ \text{D}, \\ \text{E}; \end{array}\right.$$

and therefore $2x+17=3x-4$, *by the question:* from whence, by transposition, $21=x$; so that 21, 11, 27, 22, and 37*l.* are the several required shares; amounting, in the whole, to 118*l.*

PROBLEM VII.

To find three numbers on these conditions, that the sum of the first and second shall be 15; *of the first and third* 16; *and of the second and third* 17.

If the first number be denoted by x; then it is plain, by the question, that the second will be represented by $15-x$, and the third by $16-x$. But the sum of these two last is given equal to 17; that is, $31-2x=17$; whence, by transposition, $14=2x$; and consequently $x=\frac{14}{2}=7$. Hence $15-x=8$, and $16-x=9$; which are the other two numbers required.

PROBLEM VIII.

To find that number, which being doubled, and 16 *subtracted from the product, the remainder shall as much exceed* 100 *as the required number itself is less than* 100.

The number sought being denoted by x, the double thereof will be represented by $2x$; from which subtracting 16, the remainder will be $2x-16$; and its excess above 100 equal to $2x-16-100$: therefore $2x-16-100=100-x$, *by the question;* whence $3x=216$; and consequently $x=\frac{216}{3}=72$.

PROBLEM IX.

To divide the number 75 *into two such parts that three times the greater may exceed seven times the lesser by* 15.

Let the greater part be $=x$; then will the lesser part $= 75 - x$, and we shall have $3x - 15 = 75 - x \times 7$; or, which is the same thing, $3x - 15 = 525 - 7x$: from whence $10x = 540$, and consequently $x = 54$.

PROBLEM X.

Two persons, A *and* B, *having received equal sums of money,* A *out of his paid away* 25*l. and* B *of his* 60*l. and then it appeared that* A *had just twice as much money as* B: *what money did each receive?*

Suppose x to denote the sum received by each person; then A, after paying away 25*l.* had $x - 25$; and B, after paying away 60*l.* had $x - 60$: hence $x - 25 = 2x - 120$, *by the question;* and therefore $120 - 25 = 2x - x$, that is, $95 = x$.

PROBLEM XI.

To find that number whose $\frac{1}{3}$ *part exceeds its* $\frac{1}{4}$ *part by* 12.

Let the number sought be represented by x; then will $\frac{x}{3} - \frac{x}{4} = 12$, by the conditions of the problem; which equation, by multiplying every numerator into all the denominators except its own, gives $4x - 3x = 144$, that is, $x = 144$.

PROBLEM XII.

What sum of money is that whose $\frac{1}{3}$ *part,* $\frac{1}{4}$ *part, and* $\frac{1}{5}$ *part, added together, shall amount to* 94 *pounds?*

If x be the number of pounds required, then will $\frac{x}{3} + \frac{x}{4} + \frac{x}{5} = 94$: from whence, by reduction, $20x + 15x + 12x = 94 \times 60$, that is, $47x = 94 \times 60$; and therefore $x = 2 \times 60 = 120$.

PROBLEM XIII.

In a mixture of copper, tin, and lead, one half of the whole — 16lb. *was copper; one third of the whole* — 12lb. *tin; and one fourth of the whole* + 4lb. *lead: what quantity of each was there in the composition?*

Let x denote the weight of the whole:

$$\text{then will}\left\{\begin{array}{l}\frac{x}{2}-16\\ \frac{x}{3}-12\\ \frac{x}{4}+\;4\end{array}\right\}\text{be the weight of the}\left\{\begin{array}{l}\text{copper,}\\ \text{tin,}\\ \text{lead;}\end{array}\right.$$

and, if all these be added together, we shall have $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}-24=x$, *by the question.* Hence, by reduction, $12x+8x+6x-576=24x$; therefore $2x=576$, and $x=\frac{576}{2}=288$. So that there were 128lb. of copper, 84lb. of tin, and 76lb. of lead.

PROBLEM XIV.

What sum of money is that, from which 5l. being subtracted, two-thirds of the remainder shall be 40l.?

Let x represent the required sum; then, 5 being subtracted, there will remain $x-5$; two-thirds of which will be $\overline{x-5}\times\frac{2}{3}$, or $\frac{2x-10}{3}$; and so, *by the question*, we have $\frac{2x-10}{3}=40$: whence $2x-10=120$; and $x=\frac{130}{2}=65$.

PROBLEM XV.

What number is that, which being divided by 12, the quotient, dividend, and divisor, added all together, shall amount to 64?

Let $x=$ the required number; so shall $\frac{x}{12}+x+12=64$, by the conditions of the question. Whence $x+12x=52\times12$, or $13x=624$; and consequently $x=\frac{624}{13}=48$.

PROBLEM XVI.

To find two numbers in the proportion of 2 *to* 1, *so that, if* 4 *be added to each, the two sums thence arising shall be in proportion as* 3 *to* 2.

Let x denote the lesser number; then the greater will be denoted by $2x$; and so, *by the question*, we shall have $2x + 4 : x + 4 :: 3 : 2$. From whence, as the product of the two extremes of any four proportional numbers is equal to the product of the two means (*see section* 10, *theorem* 1), we have the following equation, *viz.* $\overline{2x + 4} \times 2 = \overline{x + 4} \times 3$, that is, $4x + 8 = 3x + 12$: whence $x = 4$, and $2x = 8$: which are the two numbers that were to be found.

PROBLEM XVII.

A prize of 2000*l. was divided between two persons, whose shares therein were in proportion as* 7 *to* 9: *what was the share of each?*

If $x =$ the share of the first, then that of the second will be $2000 - x$; and we shall have $x : 2000 - x :: 7 : 9$.

Hence, by multiplying the extremes and means, $9x = 14000 - 7x$; from which x is found $= \frac{14000}{16} = 875l.$ and $2000 - x = 1125l.$

PROBLEM XVIII.

A bill of 120*l. was paid in guineas and moidores, and the number of pieces of both sorts was just* 100; *to find how many there were of each.*

If $x =$ the number of guineas, then will $100 - x$ be the number of moidores: therefore the number of shillings in the guineas being $21x$, and, in the moidores, $27 \times \overline{100 - x}$, we have $21x + 27 \times \overline{100 - x} = 120 \times 20 =$ the shillings in the whole sum: hence, by multiplication, $21x + 2700 - 27x = 2400$; and $x = \frac{300}{6} = 50$.

PROBLEM XIX.

A labourer engaged to serve 40 *days, on these conditions, that for every day he worked he was to receive* 20 *pence, but that for every day he played, or was absent, he was to forfeit* 8 *pence; now, after the* 40 *day's were expired, it was found that he had to receive, upon the whole,* 380 *pence: the question is, to find how many days of the forty he worked, and how many he played.*

Let the number expressing the days he worked be represented by x; then the number of days he played will be expressed by $40 - x$: moreover, since he was to receive 20 pence for every day he worked, the whole number of pence gained by working will be $20x$; and, for the like reason, the number of pence forfeited by playing, or being absent, will be $8 \times \overline{40 - x}$, or $320 - 8x$; which deducted from $20x$, leaves $28x - 320$, for the sum total of what he had to receive: whence we have this equation, $28x - 320 = 380$: from which $28x = 380 + 320 = 700$, and consequently $x = \frac{700}{28} = 25$, equal to the number of days he worked; therefore $40 - 25 = 15$ will be the number of days he played.

PROBLEM XX.

A farmer would mix two sorts of grain, viz. *wheat, worth* 4*s. a bushel, with rye, worth* 2*s.* 6*d. the bushel, so that the whole mixture may consist of* 100 *bushels, and be worth* 3*s.* 2*d. the bushel: now it is required to find how many bushels of each sort must be taken to make up such a mixture.*

Let the number of bushels of wheat be put $= x$, and the number of bushels of rye will be $100 - x$: but the number of bushels multiplied by the number of pence per bushel, is equal to the number of pence the whole is worth; therefore $48x$ is the whole value of the wheat, and $30 \times \overline{100 - x}$, or $3000 - 30x$, that of the rye; and, consequently, $48x + 3000 - 30x$, the sum of these two, the whole value of the mixture: which, *by the question*, is equal to 100×38, or 3800 pence: hence we have $48x + 3000 - 30x = 3800$; and therefore $x = \frac{800}{18} = 44\frac{4}{9}$, the number of bushels of wheat; whence the number of bushels of rye will be $100 - 44\frac{4}{9} = 55\frac{5}{9}$.

PROBLEM XXI.

A farmer sold to one man 30 bushels of wheat and 40 of barley, and for the whole received 270 shillings; and to another he sold 50 bushels of wheat and 30 of barley, at the same prices, and for the whole received 340 shillings: now it is required to find what each sort of grain was sold at per bushel.

Let x and y be respectively the number of shillings which a bushel of each sort was sold for; then, from the conditions of the question, we shall have these two equations, *viz.*

$$30x + 40y = 270,$$
$$50x + 30y = 340;$$

from four times the second of which subtract three times the first, so shall $110x = 550$; and consequently $x = \frac{550}{110} = 5$: moreover, by subtracting 3 times the second, from 5 times the first, you will have $110y = 330$, and therefore $y = \frac{330}{110} = 3$.

$$\text{For } \begin{cases} 30 \times 5 + 40 \times 3 = 270, \\ 50 \times 5 + 30 \times 3 = 340. \end{cases}$$

PROBLEM XXII.

A son asking his father how old he was, received the following reply: My age, says the father, 7 years ago, was just four times as great as yours at that time; but, 7 years hence, if you and I live, my age will then be only double of yours: it is required to find from hence the age of each person.

Let x represent the age of the son seven years before the question; then the age of the father, at that time, was $4x$, by the conditions of the question; and, if each of these ages be increased by 14, it is plain that $x + 14$ and $4x + 14$ will respectively express the two ages 7 years after the time in question; whence, again, by the problem, we have $4x + 14 = 2 \times \overline{x + 14}$; from which $x = 7$, and $4x = 28$; therefore $7 + 7 = 14$, and $28 + 7 = 35$, are the two ages required.

$$\text{For} \left\{ \begin{array}{l} 35 - 7 = \overline{14 - 7} \times 4, \\ 35 + 7 = \overline{14 + 7} \times 2. \end{array} \right.$$

PROBLEM XXIII.

A gentleman hired a servant for 12 *months, and agreed to allow him* 20*l. and a livery, if he staid till the year was expired; but at the end of* 8 *months the servant went away, and received* 12*l. and the livery, as a proportional part of his wages: the question is, what was the livery valued at?*

Let x be the value sought; then $20 + x$ will be the whole wages for 12 months, and $12 + x$ the part thereof which he received for 8 months.

But the wages being in the same proportion as the times in which they are earned, or become due, we therefore have, as $12 : 8 :: 20 + x : 12 + x$; whence $12 \times \overline{12 + x} = 8 \times \overline{20 + x}$, or $144 + 12x = 160 + 8x$ (by theor. 1, p. 72), consequently $12x - 8x = 160 - 144$, and $x = \frac{16}{4} = 4l.$

PROBLEM XXIV.

Four persons, A, B, C, D, *spent twenty shillings in company together; whereof* A *proposed to pay* $\frac{1}{3}$, B $\frac{1}{4}$, C $\frac{1}{5}$, *and* D $\frac{1}{6}$ *part; but, when the money came to be collected, they found it was not sufficient to answer the intended purpose: the question then is, to find how much each person must contribute, to make up the whole reckoning, supposing their several shares to be to each other in the proportion above specified.*

Let x be the share of A; then it will be, as $\frac{1}{3} : \frac{1}{4}$, or $4 : 3 :: x : \frac{3x}{4} =$ the share of B; and, as

$\frac{1}{3} : \frac{1}{5}$, or, as $5 : 3 :: x : \frac{3x}{5}$ = the share of C; also, as $\frac{1}{3} : \frac{1}{6}$, or, as $2 : 1 :: x : \frac{x}{2}$ = the share of D.

Therefore, *by the question*, $x + \frac{3x}{4} + \frac{3x}{5} + \frac{x}{2} = 20$; whence $40x + 30x + 24x + 20x = 800$, that is, $114x = 800$; and consequently $x = \frac{800}{114} = 7\frac{1}{57}$, the share of A; therefore $(\frac{3x}{4})$ that of B will be $= 5\frac{15}{57}$: that of C $(\frac{3x}{5}) = 4\frac{12}{57}$; and that of D $(\frac{x}{2}) = 3\frac{29}{57}$.

PROBLEM XXV.

A market-woman purchased a certain number of eggs at 2 a-penny, and as many at 3 a-penny, and sold them all out again at the rate of 5 for two pence, and lost four pence by so doing: what number of eggs did she buy and sell?

Let x be the number of eggs of each price, or sort; then $\frac{x}{2}$ will be the number of pence which all the first sort cost, and $\frac{x}{3}$ the price of all the second sort; but the whole price of both sorts together, at the rate of 5 for two pence, at which they were sold, will be $\frac{4x}{5}$, for as $5 : 2 :: 2x$ (the whole number of eggs) : $\frac{4x}{5}$; hence, *by the question*, $\frac{x}{2} + \frac{x}{3} - \frac{4x}{5} = 4$; whence $15x + 10x - 24x = 120$, and therefore $x = 120$.

For $\frac{120}{2} + \frac{120}{3} - \frac{240}{5} \times 2 = 60 + 40 - 96 = 4$.

PROBLEM XXVI.

A composition of copper and tin, containing 100 cubic inches, being weighed, its weight was found to be 505 ounces:

how many ounces of each metal did it contain, supposing the weight of a cubic inch of copper to be $5\frac{1}{4}$ ounces, and that of a cubic inch of tin $4\frac{1}{4}$?

Let x be the number of ounces of copper; then $505 - x$ will be the number of ounces of tin, and we shall have

$5\frac{1}{4} : 1$ (cubic inch) $: : x : \frac{x}{5\frac{1}{4}}$ inches of copper,

$4\frac{1}{4} : 1$ (cubic inch) $: : 505 - x : \frac{505 - x}{4\frac{1}{4}}$ inches of tin.

Therefore, $\frac{x}{5\frac{1}{4}} + \frac{505 - x}{4\frac{1}{4}} = 100$, *by the question*. Whence $4\frac{1}{4} \times x + 5\frac{1}{4} \times \overline{505 - x} = 5\frac{1}{4} \times 4\frac{1}{4} \times 100$, that is, $\frac{17 \times x}{4} + \frac{21 \times \overline{505 - x}}{4} \left(= \frac{21 \times 17 \times 100}{4 \times 4}\right) = \frac{21 \times 17 \times 25}{4}$; which, by rejecting the common divisor, becomes $17x + 21 \times \overline{505 - x} = 21 \times 17 \times 25 = 8925$, or $17x - 21x. = 8925 - 10605 = -1680$. From whence $x = \frac{1680}{4} = 420$; and $505 - x = 85$; which are the two numbers required.

The same otherwise.

Suppose x to be the number of solid inches of copper; then the number of inches of tin being $100 - x$, we have $5\frac{1}{4} \times x + 4\frac{1}{4} \times \overline{100 - x} = 505$, that is, $5\frac{1}{4}x + 425 - 4\frac{1}{4}x = 505$, or $x = 505 - 425 = 80$; which, multiplied by $5\frac{1}{4}$, gives 420, for the ounces of copper.

PROBLEM XXVII.

A shepherd, in time of war, fell in with a party of soldiers, who plundered him of half his flock, and half a sheep over; afterwards a second party met him, who took half what he had left, and half a sheep over; and, soon after this, a third party met him, and used him in the same manner; and then he had only five sheep left: it is required to find what number of sheep he had at first.

Let x (as usual) be the number sought; then, according to the question, the number of sheep left, after being plundered the first time, will be expressed by $\frac{x}{2} - \frac{1}{2}$, or $\frac{x-1}{2}$; the half of which is $\frac{x-1}{4}$; from whence subtracting $\frac{1}{2}$, the remainder, $(\frac{x-1}{4} - \frac{1}{2})$ $\frac{x-3}{4}$, will be the number of sheep left after being plundered the second time: in like manner, if from $\frac{x-3}{8}$ (the half of $\frac{x-3}{4}$) you again take $\frac{1}{2}$, there will remain $\frac{x-3}{8} - \frac{1}{2}$, or $\frac{x-7}{8}$, the number of sheep remaining at last. Hence we have $\frac{x-7}{8} = 5$; therefore $x - 7 = 40$, and $x = 47$.

PROBLEM XXVIII.

The difference of two numbers being given, equal to 4, *and the difference of their squares, equal to* 40; *to find the numbers.*

Let the lesser number be x; then, the difference being 4, the greater must consequently be $x + 4$, and its square $xx + 8x + 16$, from which xx, the square of the lesser being taken away, the difference is $8x + 16$: therefore $8x + 16 = 40$; which, reduced, gives $x = 3$; whence $x + 4 = 7$; therefore the two required numbers are 3 and 7.

All the problems hitherto delivered are resolved by a *numeral exegesis*, wherein the unknown quantities only are represented by letters of the alphabet; which seemed necessary, in order to strengthen the beginner's idea, at setting out, and lead him on by proper gradations: but it is not only more masterly and elegant, but also more useful, to represent the known, as well as the unknown quantities, by algebraic symbols; since from thence a general theorem is derived, whereby all other questions of the same kind may be resolved.

As an instance hereof, let the last problem be again resumed; then the given difference of the required numbers being denoted by a, the difference of their squares by b, and the lesser number by x; the greater will be $x + a$, and its square $x^2 + 2xa + a^2$; from which x^2, the square of the lesser number, being deducted, there remains $2xa + a^2 = b$: whence, if aa be subtracted from both sides, there will remain $2ax = b - aa$; this, divided by $2a$, gives $x = \frac{b}{2a} - \frac{a}{2}$; and consequently, $x + a = \frac{b}{2a} + \frac{a}{2}$. Hence it appears, that, if the difference of the squares be divided by twice the difference of the numbers, and half the difference of the numbers be subtracted from the quotient, the remainder will be the lesser number; but if half the difference of the numbers be added to the quotient, the sum will give the greater number. Thus, if the difference (a) be 4, and the difference (b) of the squares 40 (as in the case above); then $(\frac{b}{2a})$ the difference of the squares, divided by twice the difference of the numbers, will be 5; from which subtracting (2) half the difference of the numbers, there remains 3, for the lesser number sought; and by adding the said half difference, you will have $7 =$ the greater number. In the same manner, if the difference of the two numbers had been given 6, and the difference of their squares 60, the numbers themselves would have come out 2 and 8: and so of any other.

PROBLEM XXIX.

Having given the sum of two numbers, equal to 30, *and the difference of their squares, equal to* 120; *to find the numbers.*

Put $a = 30$, $b = 120$, and let x be the lesser number sought, then the greater will be $a - x$; whose square is $aa - 2ax + x^2$; from which the square of the lesser being subtracted, we have $a^2 - 2ax = b$; this reduced gives x, the lesser number, $= \frac{a}{2} - \frac{b}{2a} = 13$. Therefore the greater $(a - x)$ will be $= a - \frac{a}{2} + \frac{b}{2a} = \frac{a}{2} + \frac{b}{2a} = 17$. But if the greater number had been first made the object of our inquiry, or had been put $= x$, the lesser would have been $a - x$, and its square $a^2 - 2ax + x^2$, which, subtracted from x^2, leaves $2ax - a^2 = b$; whence $2ax = b + a^2$, and $x = \frac{b}{2a} + \frac{a}{2} = 17$, the same as before.

PROBLEM XXX.

If one agent A, *alone, can produce an effect* e, *in the time* a, *and another agent* B, *alone, in the time* b; *in how long time will they both together produce the same effect?*

Let the time sought be denoted by x, and it will be, as $a : x :: e : \frac{ex}{a}$, the part of the effect produced by A: (*theor.* 3, p. 72) also, as $b : x :: e : \frac{ex}{b}$, the part produced by B: therefore $\frac{ex}{a} + \frac{ex}{b} = e$. Divide the whole by e, and you will have $\frac{x}{a} + \frac{x}{b} = 1$; and this, reduced, gives $x = \frac{ab}{a + b}$. After the same manner, if there be three agents, A, B, and C, the time wherein they will altogether produce the given effect will come out $= \frac{abc}{ab + ac + bc}$.

Example. Suppose A, alone, can perform a piece of work in 10 days; B, alone, in 12 days; and C, alone, in 16 days: then all three together will perform the same piece of work in $4\frac{4}{59}$ days; for, in this case, a being $= 10$, $b = 12$, $c = 16$, it is plain that $\frac{abc}{ab + ac + bc}$ $\left(\frac{10 \times 12 \times 16}{10 \times 12 + 10 \times 16 + 12 \times 16}\right) = 4\frac{4}{59}$.

PROBLEM XXXI.

Two travellers, A and B, set out together from the same place, and travel both the same way; A goes 28 miles the first day, 26 the second, 24 the third, and so on, decreasing two miles every day; but B travels uniformly 20 miles every day: now it is required to find how many miles each person must travel before B comes up again with A?

Let $x =$ the number of days in which B overtakes A: then the miles travelled by B, in that time, will be $20x$; and those travelled by A, $28 + 26 + 24 + 22$, &c. continued to x terms; where the last term (*by section* 10, *theorem* 3) will be equal to $28 - 2 \times \overline{x - 1}$, or $30 - 2x$; and therefore the sum of the whole progression equal to $\overline{28 + 30 - 2x} \times \frac{1}{2}x$, or $29x - x^2$ (*by theorem* 4). Hence we have $20x = 29x - x^2$; whence $20 = 29 - x$, and $x = 9$: therefore $20 \times 9 = 180$ is the distance which was to be found.

PROBLEM XXXII.

To find three numbers, so that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third shall be equal to 62; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third, equal to 47; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third, equal to 38.

Put $a = 62$, $b = 47$, and $c = 38$, and let the numbers sought be denoted by x, y, and z; then the conditions of the problem, expressed in algebraic terms, will stand thus:

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = a,$$

$$\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = b,$$

$$\frac{x}{4} + \frac{y}{5} + \frac{z}{6} = c,$$

Which, cleared of fractions, become

$$\begin{aligned} 6x + 4y + 3z &= 12a, \\ 20x + 15y + 12z &= 60b, \\ 15x + 12y + 10z &= 60c. \end{aligned}$$

And, by subtracting the second of these equations from the quadruple of the first, in order to exterminate z, we have $4x + y = 48a - 60b$; moreover, by taking 3 times the third from 10 times the first, we have $15x + 4y = 120a - 180c$; this, subtracted from 4 times the last, leaves $x = 72a - 240b + 180c = 24$; whence $y\ (48a - 60b - 4x) = 60$, and $z\ \left(\frac{12a - 6x - 4y}{3}\right) = 120$.

$$\text{For} \begin{cases} \frac{24}{2} + \frac{60}{3} + \frac{120}{4} = 12 + 20 + 30 = 62, \\ \frac{24}{3} + \frac{60}{4} + \frac{120}{5} = 8 + 15 + 24 = 47, \\ \frac{24}{4} + \frac{60}{5} + \frac{120}{6} = 6 + 12 + 20 = 38. \end{cases}$$

PROBLEM XXXIII.

A gentleman left a sum of money to be divided among four servants, so that the share of the first was $\frac{1}{2}$ of the sum of the shares of the other three; the share of the second $\frac{1}{3}$ of the sum of the other three; and the share of the third $\frac{1}{4}$ of the sum of the other three; and it was also found that the share of the first exceeded that of the last by 14l.: the question is, what was the whole sum, and what was the share of each person?

Let the shares be represented by x, y, z, and u, respectively, and let $a = 14$; then, *by the question*, we shall have

$$\begin{aligned} x &= \frac{y + z + u}{2}, \\ y &= \frac{x + z + u}{3}, \\ z &= \frac{x + y + u}{4}, \\ u &= x - a. \end{aligned}$$

Which equations, cleared of fractions, become

$$2x = y + z + u,$$
$$3y = x + z + u,$$
$$4z = x + y + u,$$
$$u = x - a.$$

Now, if x be added to the first, y to the second, and z to the third, we shall get $(x + y + z + u) = 3x = 4y = 5z$; and from thence $z = \frac{3x}{5}$, and $y = \frac{3x}{4}$; which values being substituted in the first equation, we have $2x = \frac{3x}{4} + \frac{3x}{5} + u$, or $u = \frac{13x}{20}$; but, by the fourth equation, $u = x - a$; therefore $x - a = \frac{13x}{20}$, and $x = \frac{20a}{7} = 40$: consequently $y\ (\frac{3x}{4}) = 30$, $z\ (\frac{3x}{5}) = 24$, and $u\ (x - 14) = 26$; and the whole sum $(x + y + z + u) = 120l.$

PROBLEM XXXIV.

To find four numbers, such that the first together with half the second, may be 357 (*a*), *the second with* $\frac{1}{3}$ *of the third equal to* 476 (*b*), *the third with* $\frac{1}{4}$ *of the fourth equal to* 595 (*c*), *and the fourth with* $\frac{1}{5}$ *of the first equal to* 714 (*d*).

The required numbers being denoted by x, y, z, and u, and the conditions of the question expressed in algebraic terms, we have the four following equations:

$$x + \frac{y}{2} + a,$$
$$y + \frac{z}{3} = b,$$
$$z + \frac{u}{4} = c,$$
$$u + \frac{x}{5} = d.$$

From the first whereof we get $x = a - \frac{y}{2}$; and from the 4th, $x = 5d - 5u$; whence $a - \frac{y}{2} = 5d - 5u$, and $y = 2a - 10d + 10u$; but, by the second, $y = b - \frac{z}{3}$; therefore $2a - 10d + 10u = b - \frac{z}{3}$, and $z = 3b - 6a + 30d - 30u$; but, by the third, $z = c - \frac{u}{4}$; whence $3b - 6a + 30d - 30u = c - \frac{u}{4}$, and $12b - 24a + 120d - 120u = 4c - u$; consequently $u = \frac{12b - 24a + 120d - 4c}{119} = 676$; whence $z\ (c - \frac{u}{4}) = 426$, $y\ (= b - \frac{z}{3}) = 334$, and $x\ (= a - \frac{y}{2}) = 190$.

Otherwise.

Let the first of the required numbers be denoted by x (as above); then, the sum of the first and $\frac{1}{2}$ the second being given equal to a, it is manifest that $\frac{1}{2}$ the second must be equal to a, *minus* the first, that is $= a - x$, and therefore the second number $= 2a - 2x$: moreover, the sum of the second, and $\frac{1}{3}$ of the third, being given $= b$; it is likewise evident, that $\frac{1}{3}$ of the third must be equal to b, *minus* the second, that is $= b - 2a + 2x$, and consequently the third number itself $= 3b - 6a + 6x$: in the same manner it will appear that $\frac{1}{4}$ of the fourth number $= c - 3b + 6a - 6x$; and consequently the fourth number itself $= 4c = 12b + 24a - 24x$: whence, *by the question*, $4c - 12b + 24a - 24x + \frac{x}{5} = d$, and therefore $x = \frac{-5d + 20c - 60b + 120a}{119} = 190$; as above.

PROBLEM XXXV.

To divide the number 90 (a) *into four such parts, that if the first be increased by* 5 (b), *the second decreased by* 4 (c), *the third multiplied by* 3 (d), *and the fourth di-*

vided by 2 (e), *the result, in each case, shall be exactly the same.*

Let x, y, z, and u be the parts required; then, *by the question*, we shall have these equations, *viz.*

$$x + y + z + u = a, \text{ and}$$

$$x + b = y - c = dz = \frac{u}{e}.$$

Whence, by comparing dz with each of the three other equal values, successively, $x = dz - b$, $y = dz + c$, and $u = dez$; all which being substituted for their equals, in the first equation, we thence get $dz - b + dz + c + z + dez = a$; whence $dez + 2dz + z = a + b - c$, and $z = \frac{a + b - c}{de + 2d + 1} = 7$. Therefore x ($= dz - b$) $= 16$; y ($= dz + c$) $= 25$; and u ($= dez$) $= 42$.

PROBLEM XXXVI.

If A *and* B *together, can perform a piece of work in* 8 (a) *days;* A *and* C *together in* 9 (b) *days, and* B *and* C *in* 10 (c) *days; how many days will it take each person, alone, to perform the same work?*

Let the three numbers sought be represented by x, y, and z, respectively: then it will be, as x (days) : a (days) : : 1, the whole work, : $\frac{a}{x}$, the part thereof performed by A in a days; and, as $y : a : : 1 : \frac{a}{y}$, the part performed by B, in the same time; whence, *by the question*, $\frac{a}{x} + \frac{a}{y} = 1$ (the whole work). And, by proceeding in the very same manner, we shall have these two other equations, *viz.* $\frac{b}{x} + \frac{b}{z} = 1$, and $\frac{c}{y} + \frac{c}{z} = 1$: let the first of these three equations be divided by a, the second by b, and the third by c, and you will then have

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{a},$$
$$\frac{1}{x}+\frac{1}{z}=\frac{1}{b},$$
$$\frac{1}{y}+\frac{1}{z}=\frac{1}{c},$$

which added all together, and the sum divided by 2, give $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}$; from whence each of the three last equations being successively subtracted, we get

$$\frac{1}{z}=-\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}=\frac{-bc+ac+ab}{2abc},$$
$$\frac{1}{y}=\frac{1}{2a}-\frac{1}{2b}+\frac{1}{2c}=\frac{bc-ac+ab}{2abc},$$
$$\frac{1}{x}=\frac{1}{2a}+\frac{1}{2b}-\frac{1}{2c}=\frac{bc+ac-ab}{2abc}.$$

Hence

$$z=\frac{2abc}{-bc+ac+ab}=\frac{1440}{-90+80+72}=23\tfrac{7}{31},$$
$$y=\frac{2abc}{bc-ac+ab}=\frac{1440}{90-80+72}=17\tfrac{23}{41},$$
$$x=\frac{2abc}{bc+ac-ab}=\frac{1440}{90+80-72}=14\tfrac{34}{49}.$$

Otherwise.

Let the work performed by A in one day be denoted by x: then his work in a days will be ax, and in b days it will be bx; therefore the work of B in a days will be $1-ax$; and that of C, in b days, $1-bx$, by the conditions of the problem; whence it follows that the work of B, in one day, will be expressed by $\frac{1-ax}{a}$, and that of C, in one day, by $\frac{1-bx}{b}$; but the sum of these two last is, *by the question*, equal to $\frac{1}{c}$ part of the whole work, that is, $\frac{1}{a}+\frac{1}{b}-2x=\frac{1}{c}$;

whence $x = \frac{1}{2a} + \frac{1}{2b} - \frac{1}{2c} = \frac{bc + ac - ab}{2abc}$, equal to the work done by A in one day; by which divide 1 (the whole), and the quotient, $\frac{2abc}{bc + ac - ab}$, will give the required number of days in which he can finish the whole.

PROBLEM XXXVII.

To find three numbers on these conditions, that a times the first, b times the second, and c times the third, shall be equal to a given number p; that d times the first, e times the second, and f times the third, shall be equal to another given number q; and that g times the first, h times the second, and k times the third, shall be equal to a third given number r.

Let the three required numbers be denoted by x, y, and z, and then we shall have

$$\begin{aligned} ax + by + cz &= p, \\ dx + ey + fz &= q, \\ gx + hy + kz &= r. \end{aligned}$$

From d times the first of which subtract a times the second, and from g times the first subtract a times the third, and you will have these two new equations,

viz. $$\begin{cases} bdy - aey + cdz - afz = dp - aq, \\ bgy - ahy + cgz - akz = gp - ar; \end{cases}$$

or, which are the same,

$$\overline{bd - ae} \times y + \overline{cd - af} \times z = dp - aq,$$

and, $\overline{bg - ah} \times y + \overline{cg - ak} \times z = gp - ar$.

Multiply the first of these two equations by the coefficient of y in the second, and *vice versa*, and let the last of the two products be subtracted from the former, and you will next have $\overline{cd - af} \times \overline{bg - ah} \times z - \overline{bd - ae} \times \overline{cg - ak} \times z = \overline{bg - ah} \times \overline{dp - aq} - \overline{bd - ae} \times \overline{gp - ar}$; and therefore $z = \frac{\overline{bg - ah} \times \overline{dp - aq} - \overline{bd - ae} \times \overline{gp - ar}}{\overline{cd - af} \times \overline{bg - ah} - \overline{bd - ae} \times \overline{cg - ak}}$; whence x and y may also be found.

angle of the first into twice the second; and the third, the square of the second: from whence it is manifest, that, if the first and second terms of the square be given or expressed, not only the remaining term, but the root itself, will be found by the method above delivered.

But now, as to the ambiguity taken notice of in the third form, where $x^2 - 2ax = -b^2$, or $x^2 - 2ax + a^2 = a^2 - b^2$; the square root of the left-hand side may be either $x - a$, or $a - x$ (for either of these, squared, produces the same quantity); therefore, in the former case, $x = a + \sqrt{a^2 - b^2}$, and, in the latter, $x = a - \sqrt{a^2 - b^2}$; both which values answer the conditions of the equation. The same ambiguity would also take place in the other forms, were not the root (x) confined to a positive value.

When the highest power of the unknown quantity happens to be affected by a coefficient, the whole equation must be divided by that coefficient; and if the sign of that power be negative, all the signs must be changed before you set about to complete the square.

All equations whatever, in which there enter only two different dimensions of the unknown quantity, whereof the index of the one is just double to that of the other, are solved like quadratics, by completing the square: thus, the equation $x^4 + 2ax^2 = b$, by completing the square, will become $x^4 + 2ax^2 + a^2 = b + a^2$; whence, by extracting the root on both sides, $x^2 + a = \sqrt{b + a^2}$; therefore $x^2 = \sqrt{b + a^2} - a$, and consequently $x = \sqrt{\sqrt{b + a^2} - a}$.

These things being premised, we now proceed to the resolution of problems.

PROBLEM XXXVIII.

To find that number, to which 20 *being added, and from which* 10 *being subtracted, the square of the sum, added to twice the square of the remainder, shall be* 17475.

Let the number sought be denoted by x; then, by the conditions of the question, we shall have $\overline{x + 20}|^2 + 2 \times \overline{x - 10}|^2 = 17475$; that is, $x^2 + 40x + 400 + 2x^2 - 40x + 200 = 17475$; which, contracted, gives $3x^2 = 16875$. Hence $x^2 = 5625$; and, consequently, $x = \sqrt{5625} = 75$.

PROBLEM XXXIX.

To divide 100 *into two such parts, that, if they be multiplied together, the product shall be* 2100.

Let the excess of the greater part above (50) half the number given, be denoted by x; then $50 + x$ will be the greater part, and $50 - x$ the lesser; therefore, *by the question*, $\overline{50 + x} \times \overline{50 - x}$, or $2500 - x^2 = 2100$; whence $x^2 = 400$, and consequently $x = \sqrt{400} = 20$; therefore $50 + x = 70 =$ the greater part, and $50 - x = 30 =$ the less.

PROBLEM XL.

What two numbers are those, which are to one another in the ratio of 3 (a) *to* 5 (b), *and whose squares, added together, make* 1666 (c)?

Let the lesser of the two required numbers be x; then, $a : b :: x : \frac{bx}{a} =$ the greater; therefore, *by the question*, $x^2 + \frac{b^2x^2}{a^2} = c$; whence $a^2x^2 + b^2x^2 = a^2c$, and $x^2 = \frac{a^2c}{a^2 + b^2}$; consequently $x = \sqrt{\frac{a^2c}{a^2 + b^2}} = a\sqrt{\frac{c}{a^2 + b^2}} = 21 =$ lesser number, and $\frac{bx}{a} = 35 =$ the greater.

PROBLEM XLI.

To find two numbers, whose difference is 8, *and product* 240.

If the lesser number be denoted by x, the greater will be $x + 8$; and so, *by the question*, we shall have $x^2 + 8x$

$= 240$. Now, by completing the square, $x^2 + 8x + 16$ $(= 240 + 16) = 256$; and, by extracting the root, $x + 4 = \sqrt{256} = 16$: whence $x = 16 - 4 = 12$; and $x + 8 = 20$; which are the two numbers that were to be found.

PROBLEM XLII.

To find two numbers whose difference shall be 12, *and the sum of their squares* 1424.

Let the lesser be x, and then the greater will be $x + 12$; therefore, *by the problem*, $\overline{x + 12}|^2 + x^2 = 1424$, or $2x^2 + 24x + 144 = 1424$; this, ordered, gives $x^2 + 12x = 640$; which, by completing the square, becomes $x^2 + 12x + 36$ $(= 640 + 36) = 676$; whence, extracting the root on both sides, we have $x + 6 = (\sqrt{676})\ 26$; therefore $x = 20$, and $x + 12 = 32$, are the two numbers required.

For $\begin{cases} 32 - 20 = 12, \\ 32^2 + 20^2 = 1424. \end{cases}$

PROBLEM XLIII.

To divide 36 *into three such parts that the second may exceed the first by* 4, *and that the sum of all their squares may be* 464.

Let x be the first part, then the second will be $x + 4$; and, the sum of these two being taken from (36) the whole, we have $32 - 2x$, for the third, or remaining part; and so, *by the question*, $x^2 + \overline{x + 4}|^2 + \overline{32 - 2x}|^2 = 464$, that is, $6x^2 - 120x + 1040 = 464$; whence $6x^2 - 120x = -576$, and $x^2 - 20x = -96$. Now, by completing the square, $x^2 - 20x + 100$ $(= 100 - 96) = 4$; and, by extracting the root, $x - 10 = \mp 2$. Therefore $x = 10 \mp 2$, that is, $x = 8$, or $x = 12$; so that 8, 12, and 16 are the three numbers required.

PROBLEM XLIV.

To divide the number 100 (a) *into two such parts that their product and the difference of their squares may be equal to each other.*

Let the lesser part be denoted by x; then the greater will be $a - x$, and we shall have $\overline{a - x} \times x = \overline{a - x}|^2 - x^2$, that is, $ax - x^2 = a^2 - 2ax$; whence $x^2 - 3ax = - a^2$; and, by completing the square, $x^2 - 3ax + \frac{9a^2}{4} = (- a^2 + \frac{9a^2}{4})\ \frac{5a^2}{4}$; of which the root being extracted, there comes out $x - \frac{3a}{2} = \pm \sqrt{\frac{5aa}{4}}$, and therefore $x = \frac{3a}{2} \pm \sqrt{\frac{5aa}{4}}$. But x, by the nature of the problem, being less than a, the upper sign (+) gives x too great; so that $x = \frac{3a}{2} - \sqrt{\frac{5aa}{4}} = 38{,}19658$, &c. must be the true value required.

PROBLEM XLV.

The sum, and the sum of the squares, of two numbers being given; to find the numbers.

Let half the sum of the two numbers be denoted by a, half the sum of their squares by b, and half the difference of the numbers by x; then will the numbers themselves be represented by $a - x$, and $a + x$, and their squares by $a^2 - 2ax + x^2$, and $a^2 + 2ax + x^2$; and so we have $a^2 - 2ax + x^2 + a^2 + 2ax + x^2 = 2b$, *by the question.* Which equation, contracted and divided by 2, gives $a^2 + x^2 = b$; whence $x^2 = b - a^2$, and consequently $x = \sqrt{b - a^2}$. Therefore the numbers sought are $a - \sqrt{b - a^2}$, and $a + \sqrt{b - a^2}$.

PROBLEM XLVI.

The sum, and the sum of the cubes of two numbers being given; to find the numbers.

Let the two numbers be expressed as in the preceding problem, and let the sum of their cubes be denoted by c. Therefore will $\overline{a - x}|^3 + \overline{a + x}|^3 = c$, that is, by involution and reduction, $2a^3 + 6ax^2 = c$; whence

$6ax^2 = c - 2a^3$, $x^2 = \frac{c - 2a^3}{6a} = \frac{c}{6a} - \frac{a^2}{3}$, and $x = \sqrt{\frac{c}{6a} - \frac{a^2}{3}}$.

PROBLEM XLVII.

The sum, and the sum of the biquadrates (or 4th powers) of two numbers being given; to find the numbers.

The numbers being denoted as above, we shall here have $\overline{a - x}|^4 + \overline{a + x}|^4 = d$, that is, $2a^4 + 12a^2x^2 + 2x^4 = d$; from which, by transposition and division, $x^4 + 6a^2x^2 = \frac{1}{2}d - a^4$; and, by completing the square, $x^4 + 6a^2x^2 + 9a^4 = \frac{1}{2}d + 8a^4$; whence $x^2 + 3a^2 = \sqrt{\frac{1}{2}d + 8a^4}$; and, consequently, $x = \sqrt{-3a^2 + \sqrt{\frac{1}{2}d + 8a^4}}$.

PROBLEM XLVIII.

The sum, and the sum of the 5th powers of two numbers being given; to find the numbers.

The notation in the preceding problems being still retained, we shall have $2a^5 + 20a^3x^2 + 10ax^4 = e$; and therefore $x^4 + 2a^2x^2 = \frac{e}{10a} - \frac{a^4}{5}$; and $x^2 + a^2 = \sqrt{\frac{e}{10a} + \frac{4a^4}{5}}$; whence $x = \sqrt{\sqrt{\frac{e}{10a} + \frac{4a^4}{5}} - a^2}$.

PROBLEM XLIX.

What two numbers are those, whose product is 120 (a), *and if the greater be increased by* 8 (b), *and the lesser by* 5 (c), *the product of the two numbers thence arising shall be* 300 (d)?

If the greater number be denoted by x, and the lesser by y, we shall have
$xy = a$, and
$\overline{x + b} \times \overline{y + c} = d$, by the conditions of the question. Subtract the first of these equations from the second, and you will have $\overline{x + b} \times \overline{y + c} - xy = d - a$, that is, $cx + by + bc = d - a$; where both sides being multi-

plied by x (in order to exterminate y), we thence have $cx^2 + bxy + bcx = dx - ax$; but xy being $= a$, therefore is $bxy = ab$, and consequently, by substituting this value in the last equation, $cx^2 + ab + bcx = dx - ax$; whence $cx^2 + bcx + ax - dx = -ab$; and therefore $x^2 + bx + \frac{ax}{c} - \frac{dx}{c} = -\frac{ab}{c}$; which, by making $f = \frac{d}{c} - \frac{a}{c} - b$ $(= 28)$, will become $x^2 - fx = -\frac{ab}{c}$; hence $x^2 - fx + \frac{1}{4}f^2 = -\frac{ab}{c} + \frac{1}{4}f^2$, $x - \frac{1}{2}f = \pm\sqrt{\frac{1}{4}f^2 - \frac{ab}{c}}$, and $x = \frac{1}{2}f \pm \sqrt{\frac{1}{4}f^2 - \frac{ab}{c}} = 16$, or $= 12$; and consequently y $(\frac{a}{x}) = 10$, or $= 7{,}5$.

$$\text{For} \begin{cases} \overline{12 \times 10} = \overline{120} \\ \overline{12 + 8} \times \overline{10 + 5} = 300. \end{cases}$$

$$\text{Also} \begin{cases} \overline{16 \times 7{,}5} = \overline{120} \\ \overline{16 + 8} \times \overline{7{,}5 + 5} = 300. \end{cases}$$

PROBLEM L.

To find two numbers, such that their sum, their product, and the difference of their squares, may be all equal to one another.

The greater being denoted by x, and the lesser by y, we have $x + y = xy$, and $x + y = x^2 - y^2$: the last of these equations, divided by $x + y$, gives $1 = x - y$; whence $x = 1 + y$; this value, substituted for x in the first equation, gives $1 + 2y = y + y^2$; therefore $y^2 - y = 1$, and $y = \frac{1}{2} + \sqrt{\frac{5}{4}}$; consequently, x $(1 + y) = \frac{3}{2} + \sqrt{\frac{5}{4}}$.

PROBLEM LI.

To divide the number 100 (a) *into two such parts, that the sum of their square roots may be* 14 (b).

Let the greater part be x, and the lesser will be $a - x$; therefore, *by the problem*, $\sqrt{x} + \sqrt{a - x} = b$; and, by squaring both sides, $x + 2\sqrt{ax - xx} + a - x = bb$; whence, by transposition and division, $\sqrt{ax - xx} = \frac{bb - a}{2}$: therefore, by squaring again, $ax - xx = \frac{\overline{bb - a}|^2}{4}$, or $x^2 - ax = (-\frac{\overline{bb - a}|^2}{4}) = -\frac{b^4}{4} + \frac{b^2a}{2} - \frac{a^2}{4}$, and $x = \frac{a}{2} + \sqrt{\frac{2ab^2 - b^4}{4}} = \frac{a}{2} + \frac{b}{2}\sqrt{2a - b^2} = 64$ = the greater part; whence $a - x = 36$ = the lesser part.

PROBLEM LII.

A grazier purchased as many sheep as cost him 60*l. out of which he reserved* 15, *and sold the remainder for* 54*l. and gained two shillings a-head by them: the question is, how many sheep did he buy, and what did they cost him a-head?*

Let the number of sheep be x; then if 1200, the number of shillings which they all cost, be divided by x, the quotient, $\frac{1200}{x}$, will, it is evident, be the number of shillings which they cost him a-piece; and so the number of shillings they were sold at per head will be $\frac{1200}{x} + 2$, *by the question*; and therefore this, multiplied by $x - 15$, the number of sheep so sold, will give $1200 + 2x - \frac{18000}{x} - 30$, equal to the whole number of shillings which they were all sold for; that is, $1170 + 2x - \frac{18000}{x} = 1080$: hence we have $1170x + 2x^2 - 18000 = 1080x$, $2x^2 + 90x = 18000$, $x^2 + 45x = 9000$, and $x = \sqrt{9506.25} - 22.5 = 75$, the number of sheep; and consequently $\frac{1200}{75} = 16$ shillings, the price of each.

PROBLEM LIII.

Two country-women, A *and* B, *betwixt them, brought* 100 (c) *eggs to market; they both received the same sum for their eggs; but* A, *who had the largest and best, says to* B, *Had I brought as many eggs as you, I should have received* 18 (a) *pence for them; but, replies* B, *had I brought no more than you, I should have received only* 8 (b) *pence for mine: the question is, to find how many eggs each person had.*

If the number of eggs which A had be $= x$, the number of B's eggs will be $= c - x$; therefore, *by the problem*, it will be, $c - x : a :: x : \frac{ax}{c-x} =$ the number of pence which A received; and as $x : b :: c - x : \frac{b \times \overline{c-x}}{x} =$ the number of pence which B received: whence, again, *by the problem*, $\frac{ax}{c-x} = \frac{b \times \overline{c-x}}{x}$; and therefore $ax^2 = b \times \overline{c-x}\rceil^2 = bc^2 - 2bcx + bx^2$; which equation, ordered, gives $x^2 + \frac{2bcx}{a-b} = \frac{bc^2}{a-b}$; from whence x comes out $\left(= \sqrt{\frac{bc^2}{a-b} + \frac{b^2c^2}{\overline{a-b}\rceil^2}} - \frac{bc}{a-b}\right) = \frac{c\sqrt{ab} - bc}{a-b} = 40$. But the value of x may be otherwise more readily derived from the equation $ax^2 = b \times \overline{c-x}\rceil^2$, without the trouble of completing the square; for the square root being extracted on both sides thereof, we have $x\sqrt{a} = \overline{c-x} \times \sqrt{b}$; whence $x\sqrt{a} + x\sqrt{b} = c\sqrt{b}$, and consequently $x = \frac{c\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ $= \frac{100\sqrt{8}}{\sqrt{18}+\sqrt{8}} = \frac{100\sqrt{4}}{\sqrt{9}+\sqrt{4}} = 40$, *as before.*

PROBLEM LIV.

One bought 120 *pounds of pepper, and as many of ginger, and had one pound of ginger more for a crown than of pepper; and the whole price of the pepper exceeded that of the ginger by six crowns: how many pounds of pepper had he for a crown, and how many of ginger?*

Let the number of pounds of pepper which he had for a crown be x, and the number of pounds of ginger will be $x + 1$; moreover, the whole price of the pepper will be $\frac{120}{x}$ crowns, and that of the ginger $\frac{120}{x+1}$; therefore, *by the question*, $\frac{120}{x} - \frac{120}{x+1} = 6$; whence $120x + 120 - 120x = 6x^2 + 6x$, and therefore $x^2 + x = 20$; which, solved, gives $x = 4 =$ the pounds of pepper, and $x + 1 = 5 =$ those of ginger.

PROBLEM LV.

To find three numbers in arithmetical progression, whereof the sum of the squares shall be 1232 (a), *and the square of the mean greater than the product of the two extremes by* 16 (b).

Let the mean be denoted by x, and the common difference by y; then the numbers themselves will be $x - y$, x, and $x + y$; and so, *by the problem*, we shall have these two equations:

$\overline{x - y}|^2 + x^2 + \overline{x + y}|^2 = a$, and

$x^2 = \overline{x - y} \times \overline{x + y} + b$: these, contracted, become $3x^2 + 2y^2 = a$, and $x^2 = x^2 - y^2 + b$; from the latter whereof we get $y^2 = b = 16$; and consequently $y = \sqrt{b} = 4$; which, substituted for y in the former, gives $3x^2 + 2b = a$; whence $x^2 = \frac{a - 2b}{3}$, and therefore $x = \sqrt{\frac{a - 2b}{3}} = 20$; so that the three required numbers are 16, 20, and 24.

For $\begin{cases} 16^2 + 20^2 + 24^2 = 1232, \\ 20^2 - 16 \times 24 = 16. \end{cases}$

PROBLEM LVI.

To find two numbers whose difference shall be 10 (a), *and if* 600 (b) *be divided by each of them, the difference of the quotients shall also be equal to* 10 (a).

The lesser number being represented by x, the greater will be represented by $x + a$; and therefore, *by the problem*, $\frac{b}{x} - \frac{b}{x + a} = a$; which, freed from fractions, gives $bx + ba - bx = ax^2 + a^2x$, that is, $ba = ax^2 + a^2x$; whence, dividing by a, and completing the square, we have $x^2 + ax + \frac{1}{4}a^2 = b + \frac{1}{4}a^2$; therefore $x + \frac{1}{2}a = \sqrt{b + \frac{1}{4}a^2}$, and consequently $x = \sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a = 20$, the lesser number: whence $x + a = 30$, the greater number.

PROBLEM LVII.

To find two numbers whose sum is 80 (a), *and if they be divided alternately by each other, the sum of the quotients shall be* $3\frac{1}{3}$ (b).

If one of the numbers be x, the other will be $a - x$, and we shall therefore have $\frac{x}{a - x} + \frac{a - x}{x} = b$: which equation, brought out of fractions, becomes $x^2 + a^2 - 2ax + x^2 = abx - bx^2$; and this, by transposition, gives $2x^2 + bx^2 - 2ax - abx = -a^2$, that is, $\overline{2 + b} \times x^2 - \overline{2 + b} \times ax = -a^2$; whereof both sides being divided by $2 + b$, we have $x^2 - ax = -\frac{a^2}{2 + b}$; whence, by completing the square, $x^2 - ax + \frac{a^2}{4} = \frac{a^2}{4} - \frac{a^2}{2 + b}$; hence $x - \frac{1}{2}a = \pm\sqrt{\frac{a^2}{4} - \frac{a^2}{a + b}}$, and $x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - \frac{a^2}{2 + b}} = 60$, or $= 20$; which two are the numbers that were to be found.

PROBLEM LVIII.

To divide the number 134 (a) *into three such parts, that once the first, twice the second, and three times the* *third, added together, may be* = 278 (b), *and that the sum of the squares of all the three parts may be* = 6036 (c).

Let the three parts be denoted by x, y, and z, respectively; then, from the conditions of the problem, we shall have these three equations:

$$x + y + z = a,$$
$$x + 2y + 3z = b,$$
$$x^2 + y^2 + z^2 = c.$$

Let the first of these equations be subtracted from the second, whence $y + 2z = b - a$, or $y = b - a - 2z$; also, if the double of the first be subtracted from the second, there will come out $z - x = b - 2a$, or $x = z + 2a - b$: wherefore, if f be put $= b - a$ ($= 144$), $g = h - 2a$ ($= 10$), and for y and x, their equals $f - 2z$ and $z - g$, be substituted, our third equation, $x^2 + y^2 + z^2 = c$, will become $zz - 2gz + gg + ff - 4fz + 4zz + zz = c$; which, ordered, gives $z^2 - \frac{2f+g}{3} \times z = \frac{c - f^2 - g^2}{6}$; whence, by putting $h = \frac{2f+g}{3}$ ($= \frac{298}{3}$), and completing the square, &c. z is found $= \frac{h}{2} + \sqrt{\frac{c - f^2 - g^2}{6} + \frac{h^2}{4}}$ ($= \frac{149}{3} + \frac{1}{3}$) $= 50$: therefore y ($= f - 2z$) $= 44$, and x ($= z - g$) $= 40$.

PROBLEM LIX.

A traveller sets out from one city B, *to go to another* C, *at the same time that another traveller sets out from* C *for* B; *they both travel uniformly, and in such proportion that the former, four hours after their meeting, arrives at* C, *and the latter at* B, *in nine hours after: now, the question is to find in how many hours each person performed the journey.*

Let D be the place of meeting, and put $a = 4$, $h = 9$, and x = the number of hours they travel before they meet: then, the distances gone over, with the same uniform motion, being always to each other as the times in which they are described, we therefore have, BD : DC : : x (the time in which the first traveller goes the distance BD) : a (the time in which he goes the distance DC) ; and, for the same reason, BD : DC : : b (the time in which the second goes the distance BD) : x (the time in which he goes the distance DC): wherefore, since it appears that x is to a in the ratio of BD to DC, and b to x in the same ratio, it follows that $x : a : : b : x$; whence $x^2 = ab$, and $x = \sqrt{ab}$ $(= 6)$; therefore $a + \sqrt{ab} = 10$, and $b + \sqrt{ab} = 15$, are the two numbers required.

PROBLEM LX.

'*There are four numbers in arithmetical progression, whereof the product of the extremes is* 3250 (a), *and that of the means* 3300 (b) : *what are the numbers?*

Let the lesser extreme be represented by y, and the common difference by x; then the four required numbers will be expressed by y, $y + x$, $y + 2x$, and $y + 3x$: therefore, *by the question*, we have these two equations, *viz.*

$y \times \overline{y + 3x}$, or $y^2 + 3xy = a$, and

$\overline{y + x} \times \overline{y + 2x}$, or $y^2 + 3xy + 2x^2 = b$; whereof the former being taken from the latter, we get $2x^2 = b - a$: and from thence $x = \sqrt{\dfrac{b-a}{2}} = 5$ But, to find y from hence, we have given $y^2 + 3xy = a$ (by the first step); therefore, by completing the square, &c. $y = \sqrt{a + \dfrac{9x^2}{4}} - \dfrac{3x}{2} = 50$: and so the four numbers are 50, 55, 60, and 65.

PROBLEM LXI.

The sum (30) *and the sum of the squares* (308) *of three numbers in arithmetical progression being given; to find the numbers.*

Let the sum of the numbers be represented by $3b$, the sum of their squares by c, and the common difference by x: then, since the middle term, or number, from the nature of the progression, is $= b$, or $\frac{1}{3}$ of the whole sum, the least term, it is evident, will be expressed by $b - x$, and the greatest by $b + x$; and therefore, *by the question*, we have this equation, $\overline{b - x}|^2 + b^2 + \overline{b + x}|^2 = c$; which, contracted, gives $3b^2 + 2x^2 = c$; whence $2x^2 = c - 3b^2$, and $x = \sqrt{\frac{c - 3b^2}{2}} = 2$. Therefore 8, 10, and 12, are the three numbers sought.

PROBLEM LXII.

Having given the sum (b), *and the sum of the squares* (c), *of any given number of terms in arithmetical progression; to find the progression.*

Let the common difference be e, the first term $x + e$, and the number of terms n: then, *by the question*, we shall have

$$\overline{x + e} + \overline{x + 2e} + \overline{x + 3e} \ldots\ldots \overline{x + ne} = b, \text{ and}$$

$$\overline{x + e}|^2 + \overline{x + 2e}|^2 + \overline{x + 3e}|^2 \ldots\ldots \overline{x + ne}|^2 = c.$$

But (*by section* 10, *theo.* 4) the sum of the first of these progressions is $nx + \frac{n.\overline{n + 1}.e}{2}$: and the sum of the second (as will be shown further on) is $= nx^2 + n.\overline{n + 1}.xe + \frac{n.\overline{n + 1}.\overline{2n + 1}.e^2}{6}$: therefore our two equations will become

$$nx + \frac{n.\overline{n + 1}.e}{2} = b, \text{ and}$$

$$nx^2 + n.\overline{n + 1}.xe + \frac{n.\overline{n + 1}.\overline{2n + 1}.e^2}{6} = c.$$

Let the former whereof be squared, and the latter multiplied by n, and we shall thence have

$$n^2x^2 + n^2.\overline{n + 1}.xe + \frac{n^2.\overline{n + 1}|^2.e^2}{4} = b^2, \text{ and}$$

$$n^2x^2 + n^2.\overline{n + 1}.xe + \frac{n^2.\overline{n + 1}.\overline{2n + 1}.e^2}{6} = nc:$$

let the first of these be subtracted from the second, so shall $\frac{n^2 . \overline{n+1} . \overline{2n+1} . e^2}{6} - \frac{n^2 . \overline{n+1}\big]^2 . e^2}{4} = nc - b^2$. But $\frac{n^2 . \overline{n+1} . \overline{2n+1}}{6} - \frac{n^2 . \overline{n+1}\big]^2}{4}$ is $= n^2 . \overline{n+1} \times \overline{\frac{2n+1}{6} - \frac{n+1}{4}} = n^2 . \overline{n+1} \times \frac{8n+4-6n-6}{24} = n^2 . \overline{n+1} . \frac{\overline{2n-2}}{24} = \frac{n^2 . \overline{n+1} . \overline{n-1}}{12} = \frac{n^2 . \overline{n^2-1}}{12}$: Therefore $\frac{n^2 . \overline{n^2-1} . e^2}{12} = nc - b^2$, and $e = \sqrt{\frac{\overline{12nc-12b^2}}{n^2 \times \overline{n^2-1}}}$; whence $x \left(\frac{b}{n} - \frac{\overline{n+1} . e}{2}\right)$ is known.

Example: Let the given number of terms be 6, their sum 33, and the sum of their squares 199; then, by writing these numbers, respectively, for n, b, and c, we shall have $e = 1$; whence $x = 2$, and the required numbers 3, 4, 5, 6, 7, and 8.

PROBLEM LXIII.

Two post-boys, A *and* B, *set out, at the same time, from two cities 500 miles asunder, in order to meet each other:* A *rides 60 miles the first day, 55 the second, 50 the third, and so on, decreasing 5 miles every day: but* B *goes 40 miles the first day, 45 the second, 50 the third,* &c. *increasing 5 miles every day; now it is required to find in what number of days they will meet.*

In order to have a general solution of this problem, let the first day's distance of the post A be put $= m$, and the distance which he falls short each day of the preceding $= d$; also the first day's distance of the post B $= p$, and the distance which he gains each day $= e$; and let x be the required number of days in which they meet: then the whole distance travelled by A will be expressed by the following arithmetical progression: $m + \overline{m-d} + \overline{m-2d} + \overline{m-3d}$, &c. and that of B by

$p + \overline{p + e} + \overline{p + 2e} + \overline{p + 3e}$, &c. where each progression is to be continued to x terms. But the sum of the first of these progressions (*by sect.* 10, *theor.* 4) is $= mx - \frac{x \times \overline{x - 1} \times d}{2}$, and that of the second $= px + \frac{x \times \overline{x - 1} \times e}{2}$; therefore these two last expressions, added together, must, by the conditions of the question, be equal to 500 miles, the whole given distance; which we will call b, and then we shall have $\overline{p + m} \times x + \frac{x \times \overline{x - 1} \times \overline{e - d}}{2} = b$, or $fx + \frac{gx \times \overline{x - 1}}{2} = b$, by writing $f = p + m$, and $g = e - d$; which equation is reduced to $gx^2 - gx + 2fx = 2b$, or $x^2 - x + \frac{2fx}{g} = \frac{2b}{g}$; whence, by completing the square, &c. x comes out $= \sqrt{\frac{2b}{g} + \overline{\frac{f}{g} - \frac{1}{2}}\Big|^{2}} - \frac{f}{g} + \frac{1}{2}$. But, in the particular case proposed, the answer is more simple, and may be more easily derived from the first equation $\overline{p + m} \times x + \frac{x \times \overline{x - 1} \times \overline{e - d}}{2} = b$; for, e being $= d$, $\frac{x \times \overline{x - 1} \times \overline{e - d}}{2}$ will here entirely vanish out of the equation; and therefore x will be barely $= \frac{b}{p + m} = \frac{500}{40 + 60} = 5$. The same conclusion is also readily derived, without algebra, by the help of common arithmetic only: for, seeing the sum of the two distances travelled in the first day is 100 miles, and that the post B increases his distance, every day, by just as much as the post A decreases his, it is evident, that, between them both, they must travel 100 miles every day; therefore, if 500 be divided by 100, the quotient 5 will be the number of days in which they travel the whole 500 miles.

PROBLEM LXIV.

Two persons, A *and* B, *set out together from the same place, and travel both the same way:* A *goes* 8 *miles the first day,* 12 *the second,* 16 *the third, and so on, increasing* 4 *miles every day; but* B *goes* 1 *mile the first day,* 4 *the second,* 9 *the third, and so on, according to the square of the number of days: the question is, to find how many days each must travel before* B *comes up again with* A.

Let (4) the common difference of the progression 8, 12, 16, &c. be put $= e$, and the first term thereof *minus* the said common difference $= m$, and let the number of terms, or the days each person travels, be expressed by x: then the sum of that progression, or the number of miles which A travels, will be $x \times m + \frac{x \times \overline{x+1} \times c}{2}$ (*by sect.* 10, *theor.* 4). And (*by what follows hereafter*) the sum of the progression $1 + 4 + 9 \ldots . x^2$, or the distance travelled by B, will appear to be $\frac{x \times \overline{x+1} \times \overline{2x+1}}{6}$ therefore, *by the question,* we have $\frac{x \times \overline{x+1} \times \overline{2x+1}}{6}$ $= mx + \frac{\overline{x \times x + 1} \times e}{2}$; which, divided by x, and contracted, gives $\frac{2x^2 + 3x + 1}{6} = m + \frac{ex + e}{2}$; whence $x^2 + \frac{3x}{2} - \frac{3ex}{2} = 3m + \frac{3e}{2} - \frac{1}{2}$; and, by completing the square, $x^2 + \frac{3x}{2} - \frac{3ex}{2} + \frac{9}{16} - \frac{18e}{16} + \frac{9e^2}{16}$ $\left(= 3m + \frac{3e}{2} - \frac{1}{2} + \frac{9}{16} - \frac{18e}{16} + \frac{9e^2}{16} = \frac{48m + 1 + 6e + 9e^2}{16}\right) = \frac{48m + \overline{1 + 3e}\,]^2}{16}$; whence

$n + \frac{3}{4} - \frac{3e}{4} = \frac{\sqrt{48m + \overline{1 + 3e}\,]^2}}{4}$, and $x = \frac{\sqrt{48m + \overline{1 + 3e}\,]^2} + 3e - 3}{4} = 7$, the number of days required.

PROBLEM LXV.

The sum of the squares (a), *and the continual product* (b) *of four numbers in arithmetical progression being given; to find the numbers.*

Let the common difference be denoted by $2x$, and the lesser extreme by $y - 3x$; then, it is plain, the other three terms of the progression will be expressed by $y - x$, $y + x$, and $y + 3x$, respectively; and so, *by the question*, we have

$\overline{y - 3x}\,]^2 + \overline{y - x}\,]^2 + \overline{y + x}\,]^2 + \overline{y + 3x}\,]^2 = a$, and

$\overline{y - 3x} \times \overline{y - x} \times \overline{y + x} \times \overline{y + 3x} = b$,

that is, by reduction,

$$4y^2 + 20x^2 = a, \text{ and}$$
$$y^4 - 10y^2x^2 + 9x^4 = b;$$

from the former of which $y^2 = \frac{1}{4}a - 5x^2$: and therefore $y^4 = \frac{1}{4}a^2 - \frac{5}{2}ax^2 + 25x^4$: these values being substituted in the latter, we have $\frac{1}{16}a^2 - \frac{5}{2}ax^2 + 25x^4 - \frac{5}{2}ax^2 + 50x^4 + 9x^4 = b$, and therefore $x^4 - \frac{5ax^2}{84} = \frac{b}{84} - \frac{a^2}{16 \times 84}$; whence, by completing the square, $x^4 - \frac{5ax^2}{84} + \frac{25a^2}{4 \times 84 \times 84} \left(= \frac{b}{84} + \frac{a^2}{84 \times 84}\right) = \frac{84b + a^2}{84 \times 84}$; therefore $x^2 - \frac{5a}{2 \times 84} = \frac{\pm\sqrt{84b + a^2}}{84}$, and $x = \sqrt{\frac{5a \pm 2\sqrt{84b + a^2}}{168}}$; whence y $(= \sqrt{\frac{1}{4}a - 5x^2})$ is also known.

PROBLEM LXVI.

The difference of the means (a), *and the difference of the extremes* (b) *of four numbers in continued geometrical proportion being given; to find the numbers.*

Let the sum of the means be denoted by x; then the greater of them will be denoted by $\frac{x+a}{2}$, and the lesser by $\frac{x-a}{2}$: whence, by the nature of proportionals, it will be $\frac{x+a}{2} : \frac{x-a}{2} :: \frac{x-a}{2} : \frac{\overline{x-a}|^2}{2x+2a}$, the lesser extreme, and $\frac{x-a}{2} : \frac{x+a}{2} :: \frac{x+a}{2} : \frac{\overline{x+a}|^2}{2x-2a}$, the greater extreme: therefore, *by the problem*, we have $\frac{\overline{x+a}|^2}{2x-2a} - \frac{\overline{x-a}|^2}{2x+2a} = b$; and consequently $\overline{x+a}|^3 - \overline{x-a}|^3 = 2b \times \overline{x-a} \times \overline{x+a}$, that is, $6x^2a + 2a^3 = 2b \times \overline{x^2-a^2}$; whence $x^2 = \frac{ba^2+a^3}{b-3a}$, and consequently $x = a\sqrt{\frac{b+a}{b-3a}}$.

PROBLEM LXVII.

The sum, and the sum of the squares of three numbers in geometrical proportion being given; to find the numbers.

Let the sum of the three numbers be denoted by a, and the sum of their squares by b, and let the numbers themselves be denoted by x, y, and z: then we shall have

$$x + y + z = a,$$
$$x^2 + y^2 + z^2 = b,$$
$$\text{and } xz = y^2.$$

Transpose y in the first equation, and square both sides, so shall $x^2 + 2xz + z^2 = a^2 - 2ay + y^2$; from whence, subtracting the second equation, we have $2xz - y^2 = a^2 - 2ay + y^2 - b$: but, by the third, $2xz = 2y^2$; therefore $y^2 = a^2 - 2ay + y^2 - b$; and consequently $y = \frac{a^2 - b}{2a} = \frac{a}{2} - \frac{b}{2a}$. Now, to find x and z, y may be looked upon as known; and so, by the second equation, we have given $x^2 + z^2 = b - y^2$; from which subtracting $2xz = 2y^2$, there arises $x^2 - 2xz + z^2 = b - 3y^2$; where, the square root being extracted, we have $x - z = \sqrt{b - 3y^2}$: but, by the first equation, we have $x + z = a - y$; whence, by adding and subtracting these last equations, there results $2x = a - y + \sqrt{b - 3yy}$, and $2z = a - y - \sqrt{b - 3yy}$.

PROBLEM LXVIII.

The sum (s), *and the product* (p), *of any two numbers being given; to find the sum of the squares, cubes, biquadrates*, &c. *of those numbers.*

If the two numbers be denoted by x and y; then will

$$\left.\begin{array}{r} x + y = s \\ \text{and } xy = p \end{array}\right\} \textit{by the problem.}$$

The former of which, squared, gives $xx + 2xy + yy = s^2$; from whence subtracting the double of the latter, we have $x^2 + y^2 = s^2 - 2p$, *the sum of the squares.*

Let this equation be multiplied by $x + y = s$; so shall $x^3 + xy \times \overline{x + y} + y^3 = s^3 - 2sp$, that is, $x^3 + p \times s + y^3 = s^3 - 2sp$ (because $xy = p$, and $x + y = s$); and therefore $x^3 + y^3 = s^3 - 3sp$, *the sum of the cubes.*

Multiply, again, by $x + y = s$; then will $x^4 + xy \times \overline{x^2 + y^2} + y^4 = s^4 - 3s^2p$, or $x^4 + p \times \overline{s^2 - 2p} + y^4 = s^4 - 3s^2p$ (because $x^2 + y^2 = s^2 - 2p$). Consequently $x^4 + y^4 = s^4 - 4s^2p + 2p^2$, *the sum of the biquadrates.*

Hence the law of continuation is manifest, being such, that the sum of the next superior powers will be always obtained by multiplying the sum of the powers last found by s, and subtracting, from the product, the sum of the preceding ones multiplied by p. And thus the sum of the nth powers, expressed in a general manner, will be $s^n - ns^{n-2}p + n \cdot \frac{n-3}{2} \cdot s^{n-4}p^2 - n \cdot \frac{n-4}{2}$.

$\frac{n-5}{3} . s^{n-6} p^3 + n . \frac{n-5}{2} . \frac{n-6}{3} . \frac{n-7}{4} . s^{n-8} p^4$; &c.

PROBLEM LXIX.

The sum of the squares (a), and the excess (b) of the product above the sum of two numbers being given; to find the numbers.

Let the sum of the numbers be denoted by s, and their product by r: then the sum of their squares will be $s^2 - 2r$ (*by the last problem*), and we shall have $r - s = b$, and $s^2 - 2r = a$, whence, by adding the double of the former equation to the latter, $s^2 - 2s = a + 2b$; and consequently $s = \sqrt{a + 2b + 1} + 1$. From which r $(= b + s)$ is likewise known; and from thence the numbers themselves.

PROBLEM LXX.

The sum (a), and the sum of the squares (b) of four numbers, in geometrical progression, being given; to find the numbers.

If x and y be taken to denote the two middle numbers, the two extreme ones, by the nature of progressionals, will be truly represented by $\frac{x^2}{y}$ and $\frac{y^2}{x}$.

Put the sum of the two means $= s$, and their rectangle $= r$; so shall the sum of the two extremes $\left(\frac{xx}{y} + \frac{yy}{x}\right)$ be $= a - s$, and their rectangle also $= r$ (by the nature of the question). But (*by problem* 68) the sum of the squares of any two numbers whose sum is s, and rectangle r, will be $= ss - 2r$; and, for the same reason, the sum of the squares of our other two numbers (whose sum is $a - s$, and rectangle r) will be $= \overline{a - s}|^2 - 2r$. Therefore, by adding these aggregates of the squares of the means and extremes together, we get this equation, *viz.* $s^2 + \overline{a - s}|^2 - 4r = b$.

Moreover, from the equation, $\frac{xx}{y} + \frac{yy}{x} = a - s$, we get $x^3 + y^3 = xy \times \overline{a-s} = r \times \overline{a-s}$: but (*by the same prob. just now quoted*) $x^3 + y^3 = s^3 - 3sr$; therefore $s^3 - 3sr = ar - sr$, or $r = \frac{s^3}{2s+a}$; which value being substituted for r, in the preceding equation, we have $s^2 + \overline{a-s}|^2 - \frac{4s^3}{2s+a} = b$. This, solved, gives $s = \sqrt{\frac{aa-b}{2} + \frac{bb}{4aa}} - \frac{b}{2a}$: whence every thing else is readily found.

PROBLEM LXXI.

The sum (a) *and the sum of the squares* (b) *of five numbers, in geometrical progression, being given; to find the numbers.*

Let the three middle numbers be denoted by x, y, and z: then the two extreme ones will be $\frac{xx}{y}$ and $\frac{zz}{y}$; and therefore we shall have

$$\left.\begin{array}{l} \frac{xx}{y} + x + y + z + \frac{zz}{y} = a, \\ \frac{x^4}{y^2} + x^2 + y^2 + z^2 + \frac{z^4}{y^2} = b, \end{array}\right\} \text{by the question.}$$

Put $x + z = u$; then, by the first equation, $\frac{xx}{y} + \frac{zz}{y} = a - u - y$. Wherefore, seeing the sum of the two extremes is expressed by $a - u - y$, and their rectangle by y^2 (*see theor.* 7, *sect.* 10), the sum of their squares will (*by problem* 68) be $= \overline{a - u - y}|^2 - 2y^2$; and, in the very same manner, the sum of the squares of the two terms (x and z) adjacent to the middle one (y) will be $= u^2 - 2y^2$. Whence, by substituting these values, our equations become $\frac{u^2 - 2y^2}{y} + u + y = a$, and

$\overline{a-u-y}|^2 - 2y^2 + u^2 - 2y^2 + y^2 = b$; which, by reduction, are changed to

$$aa - 2au - 2ay + 2uu + 2uy - 2yy = b,$$
$$\text{and } ay - uu - uy + yy = 0.$$

To the former of which add the double of the latter; so shall $aa - 2au = b$; and therefore $u = \frac{a}{2} - \frac{b}{2a}$. From whence, and $yy + \overline{a-u} \times y = uu$, the value of y $\left(= \sqrt{uu + \frac{\overline{a-u}|^2}{4}} - \frac{a-u}{2}\right)$ is likewise given.

PROBLEM LXXII.

The sum (a), *the sum of the squares* (b), *and the sum of the cubes* (c), *of any four numbers in geometrical proportion being given; to find the numbers.*

Let half the sum of the two means be x, and half their difference y; also let half the sum of the two extremes be z, and half their difference v, and then the numbers themselves will be expressed thus, $z - v$, $x - y$, $x + y$, $z + v$: whence, by the conditions of the problem, we have

$$\overline{z-v} + \overline{x-y} + \overline{x+y} + \overline{z+v} = a,$$
$$\overline{z-v}|^2 + \overline{x-y}|^2 + \overline{x-y}|^2 + \overline{z+v}|^2 = b,$$
$$\overline{z-v}|^3 + \overline{x-y}|^3 + \overline{x+y}|^3 + \overline{z+v}|^3 = c,$$
$$\overline{z-v} \times \overline{z+v} = \overline{x-y} \times \overline{x+y}$$

(theor. 1, p. 72); which, contracted, are

$$2z + 2x = a,$$
$$2z^2 + 2v^2 + 2x^2 + 2y^2 = b,$$
$$2z^3 + 6zv^2 + 2x^3 + 6xy^2 = c,$$
$$z^2 - v^2 = x^2 - y^2.$$

Let $x^2 - z^2 + v^2$, the value of y^2, in the last of these equations, be substituted instead of y^2, in the two preceding ones, and we shall have

$$2z^2 + 2v^2 + 2x^2 + 2x^2 - 2z^2 + 2v^2 = b, \text{ and}$$
$$2z^3 + 6zv^2 + 2x^3 + 6x^3 - 6xz^2 + 6xv^2 = c;$$

which, abbreviated, become

$$4x^2 + 4v^2 = b, \text{ and}$$
$$2z^3 + 8x^3 - 6xz^2 + \overline{6x + 6z} \times v^2 = c.$$

Let $\frac{1}{4}b - x^2$, the value of v^2, in the former of these equations, be substituted, for its equal, in the latter, and we shall next have $2z^3 + 8x^3 - 6xz^2 + \overline{6x + 6z} \times \overline{\frac{1}{4}b - x^2} = c$; moreover, if for z, in the last equation, its equal $\frac{1}{2}a - x$ be substituted, there will come out $2 \times \overline{\frac{1}{2}a - x}\rceil^3 + 8x^3 - 6x \times \overline{\frac{1}{2}a - x}\rceil^2 + 3a \times \overline{\frac{1}{4}b - xx} = c$; that is, $6ax^2 - 3a^2x + \frac{a^3}{4} + \frac{3ab}{4} = c$; therefore $x^2 - \frac{ax}{2} = \frac{c}{6a} - \frac{b}{8} - \frac{a^2}{24}$; and, consequently, $x = \frac{a}{4} - \sqrt{\frac{c}{6a} - \frac{b}{8} + \frac{a^2}{48}}$, whence, z, v, and y are likewise known.

The same otherwise.

Let the sum of the two means $= s$, and their rectangle $= r$; so shall the sum of the two extremes $= a - s$, and their rectangle also $= r$ (*by the question*): from whence, and *prob.* 68, it is evident, that the sum of the squares of the means will be $= s^2 - 2r$, and the sum of the squares of the extremes $= \overline{a - s}\rceil^2 - 2r$; also, that the sum of the cubes of the means will be $= s^3 - 3rs$, and that of the extremes $= \overline{a - s}\rceil^3 - 3r \times \overline{a - s}$: by means whereof, and the conditions of the problem, we have given the two following equations:

viz. $s^2 + \overline{a - s}\rceil^2 - 4r = b$, or, $2s^2 - 2as - 4r = b - aa$; and $s^3 + \overline{a - s}\rceil^3 - 3ra = c$, or $3as^2 - 3a^2s - 3ar = c - a^3$: divide the former by 2, and the latter by $3a$, and then subtract the one from the other, so shall $r = \frac{aa}{6} - \frac{b}{2} + \frac{c}{3a}$, whence the value of s $\left(= \frac{a}{2} - \sqrt{\frac{b - aa}{2} + 2r + \frac{aa}{4}}\right.$, *by the first equation*) is also given, being (when substitution is made) $= \frac{a}{2} \pm \sqrt{\frac{aa}{12} - \frac{b}{2} + \frac{2c}{3a}}$.

PROBLEM LXXIII.

Having given the sum (a), and the sum of the squares (b), of any number of quantities in geometrical progression; to determine the progression.

Let the first term be denoted by x, the common ratio by z, and the given number of terms by n: then, by the conditions of the problem, we shall have

$x + xz + xz^2 + xz^3 + xz^4 \dots + xz^{n-1} = a$,

$x^2 + x^2z^2 + x^2z^4 + x^2z^6 + x^2z^8 \dots + x^2z^{2n-2} = b$.

Multiply the first equation by $1 - z$, and the second by $1 - z^2$; so shall

$$x - xz^n = a \times \overline{1 - z}, \text{ and}$$
$$x^2 - x^2z^{2n} = b \times \overline{1 - z^2}.$$

Divide the latter of these by the former; whence will be had $x + xz^n = \frac{b}{a} \times \overline{1 + z}$: let this equation and the first be now multiplied cross-wise, into each other, in order to exterminate x; so shall $a \times \overline{1 + z^n} = \frac{b}{a} \times \overline{1 + z} \times \overline{1 + z + z^2 + z^3 \dots z^{n-1}}$.

If n be an even number, put $2m = n$; then our last equation, when multiplication by $1 + z$ is actually made, will stand thus, $\frac{aa}{b} \times \overline{1 + z^{2m}} = 1 + 2z + 2z^2 \dots + 2z^{2m-2} + 2z^{2m-1} + z^{2m}$; which, divided by z^m, becomes $\frac{aa}{b} + \frac{1}{z^m} + z^m = \frac{1}{z^m} + \frac{2}{z^{m-1}} + \frac{2}{z^{m-2}} \dots + \frac{2}{z^2} + \frac{2}{z} + 2 + 2z + 2z^2 \dots + 2z^{m-2}$

$+ 2z^{m-1} + z^m$. Let s be now put $\left(= \frac{1}{z} + z\right) =$ the sum of the halves of the two terms of the series adjacent to (2) the middle one; then, the rectangle of these quantities being 1, the sum of their squares (or half the sum of the two terms of the series next to those) will be $= s^2 - 2$ (*by problem* 68); and the sum $\left(\frac{1}{z^3} + z^3\right)$ of half the two next terms to these last $= s^3 - 3s$, &c. &c.

Hence, by making $d = \frac{aa}{2b} - \frac{1}{2}$, and putting the value of $\frac{1}{z^m} + z^m$ (*as expressed in the said problem* 68) $= Q$, and then substituting above, &c. our equation becomes $dQ = 1 + s + \overline{s^2 - 2} + \overline{s^3 - 3s} + \overline{s^4 - 4s^2 + 2}$, &c. *continued to m terms;* whence the value of s may be determined.

Thus, let n, the number of terms given, be four; then m being $= 2$, Q $\left(= \frac{1}{z^2} + z^2\right)$ will be $s^2 - 2$; and our equation will, here, be $d \times \overline{s^2 - 2} = 1 + s$. If n be $= 6$, Q $\left(= \frac{1}{z^3} + z^3\right)$ will be $= s^3 - 3s$; and we shall have $d \times \overline{s^3 - 3s} = 1 + s + \overline{s^2 - 2} = s^2 + s - 1$; and so in other cases, where n is an even number.

If n be an odd number, put $2m = n - 1$; and let both sides of the equation

$$a \times \overline{1 + z^n} = \frac{b}{a} \times \overline{1 + z} \times \overline{1 + z + z^2 \ldots z^{n-1}}$$

be divided by $1 + z$; so shall

$$a \times \overline{1 - z + z^2 - z^3 \ldots - z^{n-2} + z^{n-1}} = \frac{b}{a} \times \overline{1 + z + z^2 \ldots + z^{n-1}}$$

(because $\overline{1 + z} \times \overline{1 - z + z^2 - z^3 + z^4 \ldots - z^{n-2} + z^{n-1}}$

$$= \left\{ \begin{array}{l} 1 - z + z^2 - z^3 + z^4 \ldots - z^{n-2} + z^{n-1} \quad * \\ * + z - z^2 + z^3 - z^4 \ldots + z^{n-2} - z^{n-1} + z^n \end{array} \right\} =$$

$1 + z^n)$: whence, by transposition, and substituting m, $\overline{a - \frac{b}{a}} \times \overline{1 + z^2 + z^4 \ldots + z^{2m}} = \overline{a + \frac{b}{a}} \times \overline{z + z^3 + z^5 \ldots z^{2m-1}}$; put $\frac{aa + b}{aa - b} = c$; and let the whole equation be divided by $a - \frac{b}{a} \times z^m$; then will

$$\frac{1}{z^m} + \frac{1}{z^{m-2}} + \frac{1}{z^{m-4}} \ldots + z^{m-4} + z^{m-2} + z^m = c \times \overline{\frac{1}{z^{m-1}} + \frac{1}{z^{m-3}} \ldots + z^{m-3} + z^{m-1}}.$$

Now, if m be an even number, the powers of z in the former part of the equation will be the even ones, and those in the latter the odd ones: but if m be an odd number, then *vice versa*.

In the first case, our equation may be written thus:

$$\frac{1}{z^m} + \frac{1}{z^{m-2}} \ldots + \frac{1}{z^4} + \frac{1}{z^2} + 1 + z^2 + z^4 \ldots + z^{m-2} + z^m = c \times \overline{\frac{1}{z^{m-1}} + \frac{1}{z^{m-3}} \ldots + \frac{1}{z^3} + \frac{1}{z} + z + z^3 \ldots z^{m-3} + z^{m-1}}.$$

Where, since $\frac{1}{z} + z = s$, $\frac{1}{z^2} + z^2 = s^2 - 2$, $\frac{1}{z^3} + z^3 = s^3 - 3s$, $\frac{1}{z^4} + z^4 = s^4 - 4s^2 + 2$, &c. we shall, by substituting these values in each series (proceeding from the middle both ways) have $1 + \overline{s^2 - 2} + \overline{s^4 - 4s + 2} +$ &c. $= c$ into $s + \overline{s^3 - 3s} +$ &c.

But, in the second case, where m is an odd number, and the even powers of z come into the second series, we shall, by the very same method, have $s + \overline{s^3 - 3s} + \overline{s^5 - 5s^3 + 5s} +$ &c. $= c$ into $1 + \overline{s^2 - 2s} + \overline{s^4 - 4s^2 + 2} +$ &c.

In both which cases the terms are be so far continued, that the exponent of s, in the highest of them, may be $= \frac{n-1}{2}$. Thus, if n, the given number of terms, be 3, then m $\left(\frac{n-1}{2}\right)$ being $= 1$, the equation belongs to *case* 2, and will be $s = c$, barely. If $n = 5$, then $m = 2$; and therefore $1 + s^2 - 2 = cs$, or $s^2 - 1 = cs$, *by case* 1. If n be 7, m will be 3; and so $s + \overline{s^3 - 3s} = c \times \overline{1 + s^2 - 2}$, or $s^3 - 2s = c \times \overline{s^2 - 1}$, *by case* 2. Lastly, if $n = 9$, then $m = 4$, and therefore $1 + \overline{s^2 - 2} + \overline{s^4 - 4s^2 + 2} = c \times \overline{s + s^3 - 3s}$, or $s^4 - 3s^2 + 1 = c \times \overline{s^3 - 2s}$, *by case* 1.

PROBLEM LXXIV.

Having given the sum (a), *and the sum of the cubes* (b), *of any number of terms in geometrical progression; to determine the progression.*

By retaining the notation in the last problem, and proceeding in the same manner, we here have

$a = x + xz + xz^2 \ldots + xz^{n-1} = \dfrac{xz^n - x}{z-1}$, and

$b = x^3 + x^3z^3 + x^3z^6 \ldots + x^3z^{3n-3} = \dfrac{x^3z^{3n} - x^3}{z^3 - 1}$ (*by theorem* 8, *sect.* 10).

Divide the last of these equations by the former, so shall $\dfrac{b}{a} = x^2 \times \dfrac{\overline{z-1} \times \overline{z^{3n}-1}}{\overline{z^3-1} \times \overline{z^n-1}} = x^2 \times \dfrac{z^{2n} + z^n + 1}{z^2 + z + 1}$ (because $\dfrac{z^3 - 1}{z - 1} = z^2 + z + 1$, and $\dfrac{z^{3n} - 1}{z^n - 1} = z^{2n} + z^n + 1$). Let this equation, and the square of the first, $a^2 = x^2 \times \dfrac{z^{2n} - 2z^n + 1}{z^2 - 2z + 1}$, be now multiplied, crosswise, in order to exterminate x; whence will be had $\dfrac{b}{a} \times \dfrac{z^{2n} - 2z^n + 1}{z^2 - 2z + 1} = a^2 \times \dfrac{z^{2n} + z^n + 1}{z^2 + z + 1}$: which, the numerators being divided by z^n, and the denominators by z, will stand thus,

$$b \times \frac{z^n - 2 + \frac{1}{z^n}}{z - 2 + \frac{1}{z}} = a^3 \times \frac{z^n + 1 + \frac{1}{z^n}}{z + 1 + \frac{1}{z}}.$$ Put (as before) the sum of z and $\frac{1}{z} = s$; then, their rectangle being 1, the sum of their nth powers $(z^n + \frac{1}{z^n}$ will be had in terms of s (*from problem* 68), whic sum let be denoted by S; so shall our equation becom $b \times \frac{S - 2}{s - 2} = a^3 \times \frac{S + 1}{s + 1}$: whence the value of may, in any case, be determined.

Thus if (n) the given number of terms be 3; the S (the sum of the cubes of z and $\frac{1}{z}$) being $= s^3 - 3$ we have $b \times \frac{s^3 - 3s - 2}{s - 2} = a^3 \times \frac{s^3 - 3s + 1}{s + 1}$; tha is, by division, $b \times \overline{s^2 + 2s + 1} = a^3 \times \frac{s^3 - 3s + 1}{s + 1}$.

If the number of terms be 4; then will $S = s^4 - 4s^2 + 2$; and therefore $b \times \frac{s^4 - 4s^2}{s - 2} = a^3 \times \frac{s^4 - 4s^2 + 3}{s + 1}$; which, by an actual division of the numerators, is reduced to $b \times \overline{s^3 + 2s^2} = a^3 \times \overline{s^3 - s^2 - 3s + 3}$.

Again, taking $n = 5$, we have $S = s^5 - 5s^3 + 5s$; and therefore $b \times \frac{s^5 - 5s^3 + 5s - 2}{s - 2} = a^3 \times \frac{s^5 - 5s^3 + 5s + 1}{s + 1}$ which, by division, is reduced to $b \times \overline{s^4 + 2s^3 - s^2 - 2s + 1}$ $= a^3 \times \overline{s^4 - s^3 - 4s^2 + 4s + 1}$: and so of others; where it may be observed, that the values of $S - 2$, and $S + 1$, will be always divisible by their respective denominators, except the latter, when n is either 3, or a multiple of 3.

PROBLEM LXXV.

The sum of any rank of quantities ($a + b + c + d + e +$ &c.) *being given* $= \mathrm{P}$, *the sum of all their rectangles* ($ab + ac + ad$, &c. $+ bc + bd$, &c. $+ cd$, &c.) $= \mathrm{Q}$, *the sum of all their solids* ($abc + abd + abe$, &c. $+ acd + ace$, &c. $+ bcd$, &c.) $= \mathrm{R}$, &c. &c. *it is proposed to determine the sum of the squares, cubes, biquadrates,* &c. *of those quantities.*

$$\text{Put}\begin{cases} p = b + c + d, \text{ \&c.} = \text{sum of all the quantities after the first } (a), \\ q = bc + bd + be, \text{ \&c.} + cd + ce, \text{ \&c.} = \text{the sum of their rectangles,} \\ r = bcd + bce, \text{ \&c.} + cde, \text{ \&c.} = \text{the sum of their solids.} \\ \text{\&c.} \qquad \text{\&c.} \end{cases}$$

Then will $\mathrm{P} = a + p$,
$\mathrm{Q} = pa + q$,
$\mathrm{R} = qa + r$,
$\mathrm{S} = ra + s$,
$\mathrm{T} = sa + t$, &c.

By squaring the first of these equations, we have $\mathrm{P}^2 = a^2 + 2ap + p^2$; from whence the double of the second being subtracted (in order to exterminate $2ap$), there results $\mathrm{P}^2 - 2\mathrm{Q} = a^2 + p^2 - 2q$. Where $\mathrm{P}^2 - 2\mathrm{Q}$ expresses the true sum of all the proposed squares $a^2 + b^2 + c^2 + d^2$ &c.; because, all the quantities a, b, c, d, &c. being concerned exactly alike in the original, or given equations, they must necessarily be alike concerned in the conclusions thence derived; so that if substitution for p and q were to be actually made in the equation $\mathrm{P}^2 - 2\mathrm{Q} = a^2 + p^2 - 2q$, here brought out, it is evident that no other dimensions of b, c, d, e, &c. besides the squares, can remain therein, as no dimensions of a, besides its square, have place in this equation.

In order to find the sum of all the cubes,
put A ($= \mathrm{P}$) $= a + p =$ sum of the roots,
and B ($= \mathrm{P}^2 - 2\mathrm{Q}$) $= a^2 + p^2 - 2q =$ sum of the squares; then, by multiplying the two equations together, we have $\mathrm{PB} = a^3 + pa^2 + p^2a - 2qa + p^3 - 2pq$.

From whence (to exterminate pa^2, the next inferior power of a after the highest, a^3) let $QA = pa^2 + p^2a + qa + pq$ (the product of the equations Q and A) be subducted; and there will remain $PB - QA = a^3 - 3qa + p^3 - 3pq$. To this last equation (in order to take away the next inferior power of a) add three times the equation $R = qa + r$, so shall $PB - QA + 3R = a^3 + p^3 - 3pq + 3r$. From whence it is evident that $PB - QA + 3R$ must be the required sum of all the cubes $a^3 + b^3 + c^3 + d^3$ &c. for reasons already specified with respect to the preceding case.

To determine the sum of the biquadrates, put $C = a^3 + p^3 - 3pq + 3r =$ the sum of all the cubes; then multiplying by the equation $P = a + p$ (*as before*), we get $PC = a^4 = pa^3 + p^3a - 3pqa + 3ra + p^4 - 3p^2q + 3pr$. From which (to exterminate pa^3) subtract $QB = pa^3 + p^3a - 2pqa + qa^2 + p^2q - 2q^2$ (the product of the equations Q and B): so shall $PC - QB = a^4 - qa^2 - pqa + 3ra + p^4 - 4p^2q + 3pr + 2q^2$: to this add $RA = qa^2 + pqa + ra + rp$; then will $PC - QB + RA = a^4 + 4ra + p^4 - 4p^2q + 4pr + 2q^2$: lastly, subtract $4S = 4ra + 4s$, so shall $PC - QB + RA - 4S = a^4 + p^4 - 4p^2q + 4pr + 2q^2 - 4s = D$, the sum of all the biquadrates.

In like manner (the last equation being, again, multiplied by $P = a + p$, the preceding one by $Q = pa + q$, &c. &c.) the sum of the fifth powers will be found $= PD - QC + RB - SA + 5T$: from whence, and the preceding cases, the law of continuation is manifest; the sum (F) of the sixth powers being $PE - QD + RC - SB + TA - 6U$; and the sum (G) of the seventh powers $= PF - QE + RD - SC + TB - UA + 7W$, &c. &c.

But, if you would have the several values of B, C, D, E, &c. independent of one another, in terms of the given quantities P, Q, R, S, T, &c. then will

$B = P^2 - 2Q$,

$C = P^3 - 3PQ + 3R$,

$D = P^4 - 4P^2Q + 4PR + 2Q^2 - 4S$,

$E = P^5 - 5P^3Q + 5P^2R + 5PQ^2 - 5PS - 5QR + 5T$, *&c. &c.* which values may be continued on, at pleasure, by multiplying the last by P, the last but one by — Q, the last but two by R, the last but three by — S, *&c.* and then adding all the products together; as is evident from the equations above derived.—These conclusions are of use in finding the limits of equations, and contain a demonstration of a rule, given for that purpose, by Sir *Isaac Newton*, in his *Universal Arithmetic.*

SECTION XII.

Of the Resolution of Equations of several Dimensions.

BEFORE we proceed to explain the methods of resolving cubic, biquadratic, and other higher equations, it will be requisite, in order to render that subject more clear and intelligible, to premise something concerning the origin and composition of equations.

Mr. *Harriot* has shown how equations are derived by the continued multiplication of binomial factors into each other; according to which method, supposing $x - a$, $x - b$, $x - c$, $x - d$, &c. to denote any number of such factors, the value of x is to be so taken, that some one of those factors may be equal to nothing: then, if they be multiplied continually together, their product must also be equal to nothing, that is, $\overline{x - a} \times \overline{x - b} \times \overline{x - c} \times \overline{x - d}$, &c. $= 0$: in which equation x may, it is plain, be equal to any one of the quantities, a, b, c, d, &c. since any one of these being substituted instead of x, the whole expression vanishes. Hence it appears, that an equation may have as many roots as it has dimensions, or as are expressed by the number of the factors, whereof it is supposed to be produced. Thus the quadratic equation $\overline{x - a} \times \overline{x - b} = 0$ or $x^2 \left.\begin{matrix} -a \\ -b \end{matrix}\right\} x + ab = 0$, has two roots, *a* and *b*; the cubic equation $\overline{x-a}\times\overline{x-b}\times\overline{x-c}=0$, or

$$x^3+\left.\begin{matrix}-a\\-b\\-c\end{matrix}\right\}x^2+\left.\begin{matrix}ab\\ac\\bc\end{matrix}\right\}x-abc=0,$$

has three roots, *a*, *b*, and *c*; and the biquadratic equation, $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}=0$, or

$$x^4+\left.\begin{matrix}-a\\-b\\-c\\-d\end{matrix}\right\}x^3+\left.\begin{matrix}ab+ac\\ad+bc\\bd+cd\end{matrix}\right\}x^2+\left.\begin{matrix}-abc\\-abd\\-acd\\-bcd\end{matrix}\right\}x+abcd=0,$$

has four roots, *a*, *b*, *c*, and *d*. From these equations it is observable, that the coefficient of the second term is always equal to the sum of all the roots, with contrary signs; that the coefficient of the third term is always equal to the sum of their rectangles, or of all the products that can possibly arise by combining them, two and two; that the coefficient of the fourth is equal to the sum of all their solids, or of all the products which can possibly arise, by combining them three and three; and that the last term of all is produced by multiplying all the roots continually together. And all this, it is evident, must equally hold good, when some of the roots are positive and the rest negative, due regard being had to the signs. Thus, in the cubic equation

$$\overline{x-a}\times\overline{x-b}\times\overline{x+c}=0,\text{ or }x^3+\left.\begin{matrix}-a\\-b\\+c\end{matrix}\right\}x^2+\left.\begin{matrix}+ab\\-ac\\-bc\end{matrix}\right\}x+abc=0$$

(where two of the roots, *a*, *b*, are positive, and the other, $-c$, is negative, the coefficient of the second term appears to be $-a-b+c$, and that of the third, $ab-ac-bc$, or $ab+a\times-c+b\times-c$, conformable to the preceding observations. Hence it follows, that, if one of the roots of an equation be given, the sum of all the rest will likewise be given; and that, in every equation where the second term is wanting, the sum of all the negative roots is exactly equal to that of all the positive ones; because, in this case, they mutually destroy each other. But when the coefficient of the second term is positive, then the negative roots, taken together, exceed the positive ones. But the negative roots, in any equation, may be changed to positive ones, and the positive to negative, by changing the signs of the second, fourth, and sixth terms, and so on, alternately. Thus, the foregoing equation

$$(\overline{x-a} \times \overline{x-b} \times \overline{x+c} =)\ x^3 + \left.\begin{matrix} -a \\ -b \\ +c \end{matrix}\right\} x^2 \left.\begin{matrix} +ab \\ -ac \\ -bc \end{matrix}\right\} x + abc = 0,$$

by changing the signs of the second and fourth terms, becomes

$$x^3 + \left.\begin{matrix} +a \\ +b \\ -c \end{matrix}\right\} x^2 \left.\begin{matrix} +ab \\ -ac \\ -bc \end{matrix}\right\} x - abc = 0,$$

or $\overline{x+a} \times \overline{x+b} \times \overline{x-c} = 0$; where the roots, from $+a$, $+b$, and $-c$, are now become $-a$, $-b$, and $+c$. Moreover, the negative roots may be changed to positive ones, or the positive to negative, by increasing or diminishing each, by some known quantity. Thus, in the quadratic equation $x^2 + 8x + 15 = 0$, where the two roots are -3 and -5 (and therefore both negative) if $z - 7$ be substituted for x, or, which is the same thing, if each of the roots be increased by 7, the equation will become $\overline{z-7}]^2 + 8 \times \overline{z-7} + 15 = 0$; that is, $z^2 - 6z + 8 = 0$, or $\overline{z-2} \times \overline{z-4} = 0$; where the roots are 2 and 4, and therefore both positive. This method of augmenting, or diminishing the roots of an equation is sometimes of use in preparing it for a solution, by taking away its second term; which is always performed by adding or subtracting $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{1}{4}$ part, &c. of the coefficient of the said term, according as the proposed equation rises to two, three, or four, &c. dimensions. Thus, in the quadratic equation $x^2 - 8x + 15 = 0$, let the roots be diminished by 4, that is, let $x - 4$ be put $= z$, or $x = 4 + z$: then, this value being substituted for x, the equation will become $\overline{z+4}]^2 - 8 \times \overline{z+4} + 15 = 0$, or $z^2 - 1 = 0$; in which the second term is wanting.

Likewise, the cubic equation $z^3 - az^2 + bz - c = 0$, by writing $x = -\frac{a}{3} + z$, and proceeding as above, will become $x^3 * \left.\begin{matrix} +\,b \\ -\,\frac{1}{3}a^2 \end{matrix}\right\} x + \left.\begin{matrix} +\,\frac{1}{3}ab \\ -\,c \\ -\,\frac{2}{27}a^3 \end{matrix}\right\} = 0$; and so of others.

Hence it appears, how any adfected quadratic may be reduced to a simple quadratic, and so resolved without completing the square; but this by the bye. I now proceed to the matter proposed, *viz.* the resolution of cubic, biquadratic, and other higher equations; and shall begin with showing

How to determine whether some, or all the roots of an equation be rational, and, if so, what they are.

Find all the divisors of the last term, and let them be substituted, one by one, for x in the given equation; and then, if the positive and negative terms destroy each other, the divisor so substituted is manifestly a root of the equation; but if none of the divisors succeed, then the roots, for the general part, are either irrational or impossible: for the last term, as is shown above, being always a multiple of all the roots, those roots, when rational, must necessarily be in the number of its divisors.

Examp. 1. Let the equation $x^3 - 4x^2 - 7x + 10 = 0$, be proposed; then the divisors of (10) the last term being $+1$, -1, $+2$, -2, $+5$, -5, $+10$, -10, let these quantities be successively substituted instead of x, and we shall have,

$1 - 4 - 7 + 10 = 0$, therefore 1 is a root;
$-1 - 4 + 7 + 10 = 12$, therefore -1 is no root;
$8 - 16 - 14 + 10 = -12$, therefore 2 is no root;
$-8 - 16 + 14 + 10 = 0$, therefore -2 is another root;
$125 - 100 - 35 + 10 = 0$, therefore 5 is the third root.

It sometimes happens that the divisors of the last term are very numerous; in which case, to avoid trouble, it will be convenient to transform the equation to another, wherein the divisors are fewer; and this is best effected by increasing or diminishing the roots by a unit, or some other known quantity.

Examp. 2. Let the equation propounded be $y^4 - 4y^3 - 8y + 32 = 0$; and, in order to change it to another, whose last term admits of fewer divisors, let $x + 1$ be substituted therein for y, and it will become $x^4 - 6x^2 - 16x + 21 = 0$; where the divisors of the last term are $1, -1, 3, -3, 7, -7, 21$, and -21; which being successively substituted for x, as before, we have,

$1 - 6 - 16 + 21 = 0$, therefore 1 is one of the roots;
$1 - 6 + 16 + 21 = 32$, therefore -1 is not a root;
$81 - 54 - 48 + 21 = 0$, therefore 3 is another root.

But the other two roots, without proceeding further, will appear to be impossible; for, their sum being equal to -4, the sum of the two positive roots (already found), with a contrary sign (as the second term of the equation is here wanting), their product, therefore, cannot be equal to (7) the last term divided by the product of the other roots, as it would, if all the roots were possible. However, to get an expression for these imaginary roots, let either of them be denoted by v, and the other will be denoted by $-4 - v$; which, multiplied together, give $-4v - v^2 = 7$; whence $v = -2 + \sqrt{-3}$, and consequently $-4 - v = -2 - \sqrt{-3}$. Now let each of the four roots, found above, be increased by unity, and you will have all the roots of the equation proposed.

When the equation given is a *literal one*, you may still proceed in the same manner, neglecting the known quantity and its powers, till you find what divisors succeed; for each of these, multiplied by the said quantity, will be a root of the equation. Thus, in the literal equation, $x^3 + 3ax^2 - 4a^2x - 12a^3 = 0$, the numeral divisors of the last term being $1, -1, 2, -2, 3, -3$, &c. I write these quantities, one by one, instead of x, not regarding a; and so have

$1 + 3 - 4 - 12 = -12$, therefore a is not a root;
$-1 + 3 + 4 - 12 = -6$, therefore $-a$ is no root;
$8 + 12 - 8 - 12 = 0$, therefore $2a$ is one of the roots;

$-8+12+8-12=0$, therefore $-2a$ is another root;
$27+27-12-12=30$, therefore $3a$ is not a root;
$-27+27+12-12=0$, therefore $-3a$ is the third root.

The reason of these operations is too obvious to need a further explanation. I shall here subjoin a different way, whereby the same conclusions may be derived, from Sir *Isaac Newton*'s Method of Divisors; which is thus:

Instead of the unknown quantity, substitute, successively, three or more adjacent terms of the arithmetical progression, $2, 1, 0, -1, -2$; *and, having collected all the terms of the equation into one sum, let the quantities thus resulting, together with all their divisors, be placed in a line, right against the corresponding terms of the progression* $2, 1, 0, -1, -2$; *then seek among the divisors an arithmetical progression, whose terms correspond with, or stand according to the order of the terms* $2, 1, 0, -1, -2$, *of the first progression, and whose common difference is either a unit, or some divisor of the coefficient of the highest power of the unknown quantity* (x) *in the given equation. If any such progression can be discovered, let that term of it which stands against the term* 0, *in the first progression, be divided by the common difference, and let the quotient, with the sign* $+$ *or* $-$ *prefixed, according as the progression is increasing or decreasing, be tried, as above, by substituting it for* x *in the proposed equation.*

Thus, let the proposed equation be $x^3-x^2-10x+6=0$; then, by substituting successively the terms of the progression $2, 1, 0, -1$, instead of x, there will arise $-10, -4, 6$, and 14, respectively; which, together with their divisors, being placed right against the corresponding terms of the progression $2, 1, 0, -1$, the work will stand thus:

2	-10	1 . 2 . 5 . 10	5
1	-4	1 . 2 . 4 .	4
0	$+6$	1 . 2 . 3 . 6	3*
-1	$+14$	1 . 2 . 7 . 14	2

Now, since the coefficient of the highest power (x^3) is here only divisible by a unit, I seek, among the divisors, a collateral progression whose common difference is a unit; and find the only one of this kind to be 5, 4, 3, 2; whose third term standing against the term 0 in the first progression, I therefore take and divide by unity, and then substitute the quotient, with a negative sign, instead of x, and there results $-27 - 9 + 30 + 6 = 0$; therefore -3 is, manifestly, a root of the equation.

Again, if the proposed equation were to be $2x^3 - 5x^2 + 4x - 10 = 0$, we shall, by proceeding in the same manner, have

2	-6	1 . 2 . 3 . 6	1
1	-9	1 . 3 . 9	3
0	-10	1 . 2 . 5 . 10	5*
-1	-21	1 . 3 . 7 . 21	7
-2	-54	1 . 2 . 3 . 6 . 9 &c.	9

In which case, I discover, among the divisors, the increasing arithmetical progression, 1, 3, 5, 7, 9; whose third term, 5, standing against the term 0 in the first progression, being divided by 2, the common difference, and the quotient $(\frac{5}{2})$ substituted for x, the business succeeds, the positive and negative terms destroying each other.

Moreover, if the equation $x^4 + x^3 - 29x^2 - 9x + 180 = 0$ were proposed, the work will stand as follows:

2	70	1 . 2 . 5 . 7 . 10 . 14 . 35 . 70	1	2	5	7
1	144	1 . 2 . 3 . 4 . 6 . 8 . 9 . 12 &c.	2	3	4	6
0	180	1 . 2 . 3 . 4 . 5 . 6 . 9 . 10 &c.	3	4	3	5*
-1	160	1 . 2 . 4 . 5 . 8 . 10 . 16 . 20 &c.	4	5	2	4
-2	90	1 . 2 . 3 . 5 . 6 . 9 . 10 . 15 &c.	5	6	1	3

Here are discovered no less than four progressions, whose terms differ by unity; whereof the terms corresponding to the term 0, in the first progression, are 3, 4, 3, and 5: therefore the two former progressions being ascending ones, and the two latter descending, I try the quantities $+3$, $+4$, -3, -5, one by one, and find that they all succeed.

And after the same manner we may proceed in other cases; but, in order to try whether any quantity thus found be a true root, we may, instead of substituting for x, divide the whole equation by that quantity connected to x, with a contrary sign; for, if the division terminate without a remainder, the said quantity is, manifestly, a root of the equation.

Thus, in the last example, where the equation is $x^4 + x^3 - 29x^2 - 9x + 180 = 0$, the numbers to be tried being $+3$, $+4$, -3, and -5, I first take -3 and join it to x, and then divide the whole equation, $x^4 + x^3 - 29x^2 - 9x + 180$ $(= 0)$, by $x - 3$, the quantity thence arising, and find the quotient to come out $x^3 + 4x^2 - 17x - 60$, exactly. Therefore $+3$ is one of the roots.

Again, in order to try $+4$, the second number, I divide the quotient thus found, by $x - 4$, and there comes out $x^2 + 8x + 15$; therefore $+4$ is another root: lastly, I try -3, by dividing the last quotient by $x + 3$, and find it also to succeed, the quotient being $x + 5$. See the operation at large.

$$\begin{array}{lllllll}
x-3) & x^4 & +x^3 & -29x^2 & -9x & +180 & (x^3+4x^2-17x-60 \\
 & x^4 & -3x^3 & & & & \\ \cline{2-4}
 & & +4x^3 & -29x^2 & & & \\
 & & +4x^3 & -12x^2 & & & \\ \cline{2-5}
 & & & -17x^2 & -9x & & \\
 & & & -17x^2 & +51x & & \\ \cline{4-6}
 & & & & -60x & +180 & \\
 & & & & -60x & +180 & \\ \cline{5-6}
 & & & & 0 & 0 & \\ \cline{5-6}
\end{array}$$

$$\begin{array}{llllll}
x-4) & x^3 & +4x^2 & -17x & -60 & (x^2+8x+15 \\
 & x^3 & -4x^2 & & & \\ \cline{2-3}
 & & +8x^2 & -17x & & \\
 & & +8x^2 & -32x & & \\ \cline{2-5}
 & & & +15x & -60 & \\
 & & & +15x & -60 & \\ \cline{4-5}
 & & & 0 & 0 & \\ \cline{4-5}
\end{array}$$

$$\begin{array}{r|l} x+3) & x^2+8x+15\;(x+5 \\ & \underline{x^2+3x} \\ & \quad+5x+15 \\ & \quad\underline{+5x+15} \\ & \qquad 0 \quad 0 \end{array}$$

As another instance hereof, let there be proposed the equation $2x^3 - 3x^2 + 16x - 24 = 0$; then, expounding x by 2, 1, 0, and -1, successively, and proceeding as in the foregoing examples, we have

2	$+12$	1 . 2 . 3 . 4 . 6 . 12	$+2$	-1
1	-9	1 . 3 . 9	$+3$	$+1$
0	-24	1 . 2 . 3 . 4 . 6 . 8 &c.	$+4$	$+3$*
-1	-45	1 . 3 . 5 . 9 . 15 . 45	$+5$	$+5$

Therefore, the quantities to be tried being 4 and $\frac{3}{2}$, I first attempt the division by $x - 4$; which does not answer: but, trying $x - \frac{3}{2}$, or (its double) $2x - 3$, I find it to succeed, the quotient being $x^2 + 8$, exactly.

The reason why the divisors, thus found, do not always succeed, is, because the first progression, 2, 1, 0, -1 is not continued far enough to know whether the corresponding progression may not break off, after a certain number of terms; which it never can do when the business succeeds. Thus, in the last example, where we had two different progressions resulting, had the operation, or series, 2, 1, 0, -1, been continued only two terms farther, you would have found the first of those progressions to fail; whereas, on the contrary, the last (by which the business succeeds) will hold, carry on the progression, 2, 1, 0, -1, as far as you will. The grounds of which, as well as of the whole method, upon which the foregoing observations are founded, may be explained in the following manner.

Let there be assumed any equation, as $ax^4 + bx^3 + cx^2 + dx + e = 0$, wherein a, b, c, d, and e, represent any whole numbers, positive or negative, and let $px + q$ denote any binomial divisor by which the said expression

$ax^4 + bx^3 + cx^2 + dx + e$ is divisible, and let the quotient thence arising be represented by $rx^3 + sx^2 + tx + v$, or, which is the same in effect, let $ax^4 + bx^3 + cx^2 + dx + e = \overline{px + q} \times \overline{rx^3 + sx^2 + tx + v}$. This being premised, suppose x to be now, successively, expounded by the terms of the arithmetical progression 2, 1, 0, -1, -2 (*as above*); and then the corresponding values of our divisor $px + q$, will, it is manifest, be expounded by $2p + q$, $p + q$, q, $-p + q$, and $-2p + q$, respectively; which also constitute an arithmetical progression, whose common difference is p; which common difference (p) must be some divisor of the coefficient (a) of the first term, otherwise the division could not succeed, that is, p could not be had in a without a remainder.

Hence it appears that the binomial divisor, by which an expression of several dimensions is divisible, must always vary as x varies, so as to be, successively, expressed by the terms of an arithmetical progression, whose common difference is some divisor of the first, or highest term of that expression.

It also appears that the said common difference is always the coefficient of the first term of the general divisor; and that the term (q) of the progression, which arises by taking $x = 0$, is the second term. Therefore, whenever, by proceeding according to the method above prescribed, a progression is found, answering to the conditions here specified, the terms of that progression are to be considered only as so many successive values of some general divisor, as $px + q$. Whence the reason of the whole process is manifest.

After the same manner we may proceed to the invention of trinomial divisors, or divisors of two dimensions: for, let $mx^2 + px + q$, be any quantity of this kind, wherein m, p, and q represent whole numbers positive or negative, and let the terms of the progression 3, 2, 1, 0, -1, -2, -3, be written therein, one by one, instead of x; whence it will become $9m + 3p + q$, $4m + 2p + q$, $m + p + q$, q, $m - p + q$, $4m - 2p + q$, and $9m - 3p + q$, respectively; where m must be some divisor of the coefficient of the first term of the given expression; otherwise, the division could not succeed. Hence it appears,

1°, That the coefficient (m) of the first term of the divisor must always be some numeral divisor of the coefficient of the first term of the proposed expression.

2°, That the product of that coefficient by the square of each of the terms of the assumed progression, 3, 2, 1, 0, — 1, — 2, — 3, being subtracted from the corresponding value of the general divisor, the remainders ($3p + q$, $2p + q$, $p + q$, q, $-p + q$, $-2p + q$, $-3p + q$) will be a series of quantities in arithmetical progression, whose common difference is the coefficient of the second term of the divisor.

3°. And that the term (q) of this progression, which arises by taking $x = 0$, will always be the third, or last term of the said divisor. From whence we have the following rule. *Instead of x in the quantity proposed, substitute, successively, four or more adjacent terms of the progression* 3, 2, 1, 0, — 1, — 2, — 3; *and from all the several divisors of each of the numbers thus resulting, subtract the squares of the corresponding terms of that progression multiplied by some numeral divisor of the highest term of the quantity proposed, and set down the remainders right against the corresponding terms of the progression* 3, 2, 1, 0, — 1, — 2, — 3; *and then seek out a collateral progression which runs through these remainders; which being found, let a trinomial be assumed, whereof the coefficient of the first term is the foresaid numeral divisor; that of the second term, the common difference of this collateral progression; and whereof the third term is equal to that term of the said progression which arises by taking $x = 0$; and the expression so assumed will be the divisor to be tried. But it is to be observed, that the second term must have a negative or positive sign, according as the progression, found among the divisors, is an increasing or a decreasing one.*

Thus, let the quantity proposed be $x^4 - x^3 - 5x^2 + 12x - 6$; and then, by substituting 3, 2, 1, 0, — 1, — 2, successively, instead of x, the numbers resulting will be 39, 6, 1, — 6, — 21, and — 26, respectively; which, together with all their divisors, both positive and negative, I place right against the corresponding terms of the progression 3, 2, 1, 0, — 1, — 2, in the following manner:

3	39. 13.3 . 1 . — 1 . — 3 . — 13 . — 39
2	6. 3.2 . 1 . — 1 . — 2 . — 3 . — 6
1	1 . — 1
0	6. 3.2 . 1 . — 1 . — 2 . — 3 . — 6
— 1	21. 7.3 . 1 . — 1 . — 3 . — 7 . — 21
— 2	26. 13.2 . 1 . — 1 . — 2 . — 13 . — 26

Then, from each of these divisors I subtract the square of the corresponding term of the first progression multiplied by unity (as being the only numeral divisor of the first term), and the work stands thus:

3	30. 4.—6.—8.—10.—12.—22.—48	+ 4	— 6
2	2.—1.—2.—3.— 5.— 6.— 7.—10	+ 2	— 3
1	0.—2. ————————	+ 0	+ 0
0	6. 3. 2. 1.— 1.— 2.— 3.— 6	— 2	+ 3*
—1	20. 6. 2. 0.— 2.— 4.— 8.—22	— 4	+ 6
—2	22. 9.—2.—3.— 5.— 6.—17.—30	— 6	+ 9

Here I discover, among the remainders, two collateral progressions, *viz.* 4, 2, 0, — 2, — 4, — 6, and — 6, — 3, 0, + 3, + 6, + 9; therefore the quantity to be tried is either $x^2 + 2x - 2$, or $x^2 - 3x + 3$; by both of which the business succeeds.

This invention of trinomial divisors is sometimes of use in finding out the roots of an equation when they are irrational, or imaginary. Thus, let the equation given be $x^4 - 4x^3 + 5x^2 - 4x - 1 = 0$; and let x be successively expounded by the terms of the progression 3; 2, 1, 0, and the numbers resulting will be 7, — 3, — 1, and 1; which, together with their divisors, being ordered according to the preceding directions, the operation will stand as follows:

3	7 . 1 .—1 .—7	—2 .—8 .—10—16	—2	—8
2	3 . 1 .—1 .—3	—1 .—3 .— 5— 7	—1	—5
1	1 .—1 * *	0 .—2 * *	0	—2
0	1 .—1 * *	1 .—1 * *	+1	+1*

Here we have two progressions, — 2, — 1, 0, 1; and — 8, — 5, — 2, 1; therefore the quantity to be tried is either $x^2 - x + 1$, or $x^2 - 3x + 1$; but I take the first, and having divided $x^4 - 4x^3 + 5x^2 - 4x + 1$ thereby, find it to succeed, the quotient coming out $x^2 - 3x + 1$, exactly. Therefore $x^4 - 4x^3 + 5x^2 - 4x + 1$ being universally equal to $\overline{x^2 - x + 1} \times \overline{x^2 - 3x + 1}$, let $x^2 - x + 1$ be taken $= 0$, and also $x^2 - 3x + 1 = 0$; from the former of which equations we have $x = \frac{1}{2} \pm \sqrt{-\frac{3}{4}}$; and from the latter $x = \frac{3}{2} \pm \sqrt{\frac{5}{4}}$. Therefore the four roots of the given equation are $\frac{1}{2} + \sqrt{-\frac{3}{4}}$, $\frac{1}{2} - \sqrt{-\frac{3}{4}}$, $\frac{3}{2} + \sqrt{\frac{5}{4}}$ and $\frac{3}{2} - \sqrt{\frac{5}{4}}$; whereof the two last are irrational and the two first imaginary. And in the same manner, the roots of a literal equation, as $z^4 - 4az^3 + 5a^2z^2 - 4a^3z + a^4 = 0$, where the terms are homogeneous, may be derived; for, let the roots be divided by a, that is, let x be put $= \frac{z}{a}$, or $ax = z$; and then, this value being substituted for z, the equation will become $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$; from which x will be found, as above; whence z $(= ax)$ is also known.

Having treated largely of the manner of managing such equations as can be resolved into rational factors, whether binomials or trinomials, I come now to explain the more general methods, by which the roots of equations, of several dimensions, are determined; and shall begin with

The Resolution of cubic Equations, according to Cardan.

If the given equation have all its terms, the second term must be taken away, as has been taught at the beginning of this section; and then the equation will be reduced to this form: *viz.* $x^3 + ax = b$; where a and b represent given quantities. Put $x = y + z$; and then, this value being substituted for x, our equation becomes $y^3 + 3y^2z + 3yz^2 + z^3 + a \times \overline{y + z} = b$, or $y^3 + z^3 + 3yz \times \overline{y + z} + a \times \overline{y + z} = b$. Assume, now, $3yz = -a$; so shall the terms $3yz \times \overline{y + z}$ and $a \times \overline{y + z}$ destroy each other, and our equation will be reduced to $y^3 + z^3 = b$. From the square of which, let four times the cube of the equation $yz = -\frac{1}{3}a$ be subtracted, and we shall have $y^6 - 2y^3z^3 + z^6 = b^2 + \frac{4a^3}{27}$; and therefore, by extracting the square root, on both sides, $y^3 - z^3 = \sqrt{b^2 + \frac{4a^3}{27}}$; which added to, and subtracted from $y^3 + z^3 = b$, gives $2y^3 = b + \sqrt{b^2 + \frac{4a^3}{27}}$, and $2z^3 = b - \sqrt{b^2 + \frac{4a^3}{27}}$: hence $y = \overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}}$, and $z = \overline{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}}$; and consequently x $(= y + z) = \overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}} + \overline{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}}$: which is *Cardan*'s theorem: but the same thing may be exhibited in a manner rather more commodious for practice by substituting for the second term its equal $\frac{-\frac{1}{3}a}{\overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}}}$ $(= \frac{-\frac{1}{3}a}{y} = z$, because $yz = -\frac{1}{3}a)$. And, this being done, our theorem stands thus:

$$x = \overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}} - \frac{\frac{1}{3}a}{\overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}}}.$$

Example 1. Let the equation $y^3 + 3y^2 + 9y = 13$ be propounded; and, in order to destroy the second term thereof, let $x - 1$ be put $= y$; so shall $\overline{x - 1}\big|^3 + 3 \times \overline{x - 1}\big|^2 + 9 \times \overline{x - 1} = 13$, or $x^3 + 6x = 20$; therefore, in this case, a being $= 6$, and $b = 20$, we have x $\left(\overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}} - \frac{\frac{1}{3}a}{\overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}}}\right) =$

$\overline{10 + \sqrt{100 + 8}}\Big]^{\frac{1}{3}} - \dfrac{2}{\overline{10 + \sqrt{100 + 8}}\Big]^{\frac{1}{3}}} = \overline{20,3923}\Big]^{\frac{1}{3}}$ $\dfrac{2}{\overline{20,3923}\Big]^{\frac{1}{3}}} = 2.732 - .732 = 2$; and consequently y $(= x - 1) = 1$.

Exam. 2. If the equation given be $y^6 - 3y^4 - 2y^2 - 8 = 0$; then, by writing $x + 1$ for y^2, it will become $\overline{x + 1}\big]^3 - 3 \times \overline{x + 1}\big]^2 - 2 \times \overline{x + 1} - 8 = 0$, or $x^3 - 5x = 12$: therefore, a being $= -5$, and $b = 12$, x will here be equal to $\overline{6 + \sqrt{36 - \frac{125}{27}}}\Big]^{\frac{1}{3}} - \dfrac{-\frac{5}{3}}{\overline{6 + \sqrt{36 - \frac{125}{47}}}\Big]^{\frac{1}{3}}} = \overline{6 + 5,6009}\Big]^{\frac{1}{3}} + \dfrac{1,6666, \text{\&c.}}{\overline{6 + 5,6009}\Big]^{\frac{1}{3}}}$ $= 2,26376 + ,73624 = 3$; and consequently y^2 $(= x + 1) = 4$: which is the only possible value of y^2 in the given equation. And it will be proper to take notice here, that this method is only of use in cases where two of the three roots are impossible (except when they are equal); for $\frac{b^2}{4} + \frac{a^3}{27}$ being, in all other cases, a negative quantity, its square root is manifestly impossible.

I shall now give the investigation of the same general theorem, for the solution of cubics, by a different method; which is also applicable to other higher equations.

Supposing, then, the sum of two numbers, z and y, to be denoted by s, and their product (zy) by p, it will appear (*from prob.* 68, *p.* 119) that the sum of their cubes $(z^3 + y^3)$ will be truly expressed by $s^3 - 3ps$.

If, therefore, $z^3 + y^3$ be assumed $= b$, we shall also have $s^3 - 3ps = b$: but, zy being $= p$, or $y = \frac{p}{z}$, our first equation $z^3 + y^3 = b$, will become $z^3 + \frac{p^3}{z^3} = b$; from which, by completing the square, &c. z is found

$= \overline{\tfrac{1}{2}b + \surd\,\overline{\tfrac{1}{4}bb - p^3}}\,\Big|^{\frac{1}{3}}$: whence y $(= \frac{p}{z})$ is given $=$ $\dfrac{p}{\overline{\tfrac{1}{2}b + \surd\,\overline{\tfrac{1}{4}bb - p^3}}\,\Big|^{\frac{1}{3}}}$; and consequently s $(= z + y)$ $= \overline{\tfrac{1}{2}b + \surd\,\overline{\tfrac{1}{4}bb - p^3}}\,\Big|^{\frac{1}{3}} + \dfrac{p}{\overline{\tfrac{1}{2}b + \surd\,\overline{\tfrac{1}{4}bb - p^3}}\,\Big|^{\frac{1}{3}}}$: which is, evidently, the true root of the equation $s^3 - 3ps = b$. From whence the root of the equation $x^3 + ax = b$, wherein the second term is positive, will be given, by writing x for s, and $\frac{1}{3}a$ for $-p$; whence x is found $= \overline{\tfrac{1}{2}b + \sqrt{\dfrac{bb}{4} + \dfrac{a^3}{27}}}\,\Bigg|^{\frac{1}{3}} - \dfrac{\tfrac{1}{3}a}{\overline{\tfrac{1}{2}b + \sqrt{\dfrac{bb}{4} + \dfrac{a^3}{27}}}\,\Bigg|^{\frac{1}{3}}}$, *the same as before.*

In like manner, if things be supposed as above, and there be now given $z^5 + y^5 = b$; then, *by the problem there referred to*, we likewise have $s^5 - 5ps^3 + 5p^2s = b$. But the first equation, by substituting $\frac{p}{z}$ for its equal y, becomes $z^5 + \frac{p^5}{z^5} = b$: whence $z^{10} - bz^5 = -p^5$, $z^5 = \frac{1}{2}b + \surd\,\overline{\tfrac{1}{4}bb - p^5}$, and $z = \overline{\tfrac{1}{2}b + \surd\,\overline{\tfrac{1}{4}bb - p^5}}\,\Big|^{\frac{1}{5}}$; and consequently s $(= z + y = z + \frac{p}{z}) =$ $\overline{\tfrac{1}{2}b + \surd\,\overline{\tfrac{1}{4}bb - p^5}}\,\Big|^{\frac{1}{5}} + \dfrac{p}{\overline{\tfrac{1}{2}b + \surd\,\overline{\tfrac{1}{4}bb - p^5}}\,\Big|^{\frac{1}{5}}}$ $=$ the true root of the equation $s^5 - 5ps^3 + 5p^2s = b$. Which by substituting x for s, and $-\frac{a}{5}$ for p, gives $x =$ $\overline{\tfrac{1}{2}b + \surd\,\overline{\tfrac{1}{4}bb + \overline{\tfrac{1}{5}a}\,\big|^{5}}}\,\Big|^{\frac{1}{5}} - \dfrac{\tfrac{1}{5}a}{\overline{\tfrac{1}{2}b + \surd\,\overline{\tfrac{1}{4}bb + \overline{\tfrac{1}{5}a}\,\big|^{5}}}\,\Big|^{\frac{1}{5}}}$, for the true root of the equation $x^5 + ax^3 + \frac{1}{5}a^2x = b$.

Generally, supposing $z^n + y^n = b$, or $z^n + \frac{p^n}{z^n} = b$ (because $y = \frac{p}{z}$), we have $z^{2n} - bz^n = - p^n$; whence $z^n = \frac{1}{2}b + \sqrt{\frac{1}{4}bb - p^n}$, and $z = \left.\overline{\frac{1}{2}b + \sqrt{\frac{1}{4}bb - p^n}}\right|^{\frac{1}{n}}$: therefore $s\ (z + y) = z + \frac{p}{z} = \left.\overline{\frac{1}{2}b + \sqrt{\frac{1}{4}bb - p^n}}\right|^{\frac{1}{n}} + \frac{p}{\left.\overline{\frac{1}{2}b + \sqrt{\frac{1}{4}bb + p^n}}\right|^{\frac{1}{7}}}$; which is the true root of the equation $s^n - nps^{n-2} + n \cdot \frac{n-3}{2} \cdot p^2s^{n-4} - n \cdot \frac{n-4}{2} \cdot \frac{n-5}{3} \cdot p^3s^{n-6} + n \cdot \frac{n-5}{2} \cdot \frac{n-6}{3} \cdot \frac{n-7}{4} \cdot p^4s^{n-8}$ — &c. $(= z^n + y^n) = b$.

This equation, by writing x for s, and $\frac{a}{n}$ for $- p$, becomes $x^n + ax^{n-2} + \frac{n-3}{2n} \cdot a^2x^{n-4} + \frac{n-4}{2n} \cdot \frac{n-5}{3n} \cdot a^3x^{n-6} + \frac{n-5}{2n} \cdot \frac{n-6}{3n} \cdot \frac{n-7}{4n} \cdot a^4x^{n-8}$, &c. $= b$; and its root $x = \left.\overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^n}{n^n}}}\right|^{\frac{1}{n}} - \frac{a}{n \times \left.\overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^n}{n^n}}}\right|^{\frac{1}{n}}}$. Wherein the two preceding theorems are included, with innumerable others of the same kind; but as every one of them, except the first, requires a particular relation of the coefficients, seldom occurring in the resolution of problems, I shall take no further notice of them here, but proceed to

The Resolution of Biquadratic Equations, according to Des Cartes.

Here the second term is to be destroyed as in the solution of cubics; which being done, the given equation will be reduced to this form, $x^4 + ax^2 + bx + c = 0$; wherein a, b, and c may represent any quantities whatever, positive or negative. Assume $\overline{x^2 + px + q} \times \overline{x^2 + rx + s} = x^4 + ax^2 + bx + c$; or, which is the same thing, let the biquadratic be considered as produced by the multiplication of the two quadratics $x^2 + px + q = 0$, and $x^2 + rx + s = 0$: then, these last being actually multiplied into each other, we shall have $x^4 + ax^2 + bx + c = x^4 + \left.\begin{matrix}p\\r\end{matrix}\right\} x^3 + \left.\begin{matrix}s\\q\\pr\end{matrix}\right\} x^2 + \left.\begin{matrix}ps\\qr\end{matrix}\right\} x + qs$; whence, by equating the homologous terms (in order to determine the value of the assumed coefficients p, q, r, and s) we have $p + r = 0$, $s + q + pr = a$, $ps + qr = b$, and $qs = c$; from the first of which $r = -p$; from the second $s + q\ (= a - pr) = a + p^2$; and from the third $s - q = \frac{b}{p}$. Now, by subtracting the square of the last of these from that of the precedent, we have $4qs = a^2 + 2ap^2 + p^4 - \frac{bb}{pp}$, that is, $4c = a^2 + 2ap^2 + p^4 - \frac{bb}{pp}$ (because $qs = c$); and therefore $p^6 + 2ap^4 \left.\begin{matrix}+a^2\\-4c\end{matrix}\right\} p^2 = b^2$; from which p will be determined, as in example the second, of the solution of cubics: Whence $s\ (= \frac{1}{2}a + \frac{1}{2}p^2 + \frac{b}{2p})$, and $q\ (= \frac{1}{2}a + \frac{1}{2}p^2 - \frac{b}{2p})$ are also known. And, by extracting the roots of the two assumed quadratics $x^2 + px + q = 0$, and

$x^2 + rx + s = 0$, we have x, in the one, $= -\frac{p}{2} \pm \sqrt{\frac{pp}{4} - q}$; and, in the other, $= -\frac{r}{2} \pm \sqrt{\frac{rr}{4} - s} = \frac{p}{2} \pm \sqrt{\frac{pp}{4} - s}$, because $r = -p$. Therefore the four roots of the biquadratic, $x^4 + ax^2 + bx + c = 0$, are $\frac{p}{2} + \sqrt{\frac{pp}{4} - s}$, $\frac{p}{2} - \sqrt{\frac{pp}{4} - s}$, $-\frac{p}{2} + \sqrt{\frac{pp}{4} - q}$, and $-\frac{p}{2} - \sqrt{\frac{pp}{4} - q}$.

EXAMPLE.

Let the equation propounded be $y^4 - 4y^3 - 8y + 32 = 0$; then, to take away the second term thereof, let $x + 1 = y$; whence, by substitution, $x^4 * - 6x^2 - 16x + 21 = 0$; which being compared with the general equation, $x^4 * + ax^2 + bx + c = 0$, we here have $a = -6$, $b = -16$, and $c = 21$; and consequently $p^6 - 12p^4 - 48p^2$ $(= p^6 + 2ap^4 \left.\begin{matrix}+a^2\\-4c\end{matrix}\right\} p^2) = 256$ $(= b^2)$. Now, to destroy the second term of this last equation also, make $z + 4 = p^2$; and then, this value being substituted, you will have $z^3 - 96z = 576$; whence, by the method above explained, z will be found $\left(= \overline{288 + \sqrt{\overline{288}^2 - \overline{32}^3}}^{\,\frac{1}{3}} + \frac{32}{\overline{288 + \sqrt{\overline{288}^2 - \overline{32}^3}}^{\,\frac{1}{3}}}\right) = 12$. Therefore p $(= \sqrt{z + 4})$ is $= 4$, s $(= \frac{a}{2} + \frac{p^2}{2} + \frac{b}{2p}) = 3$, and q $(= \frac{a}{2} + \frac{p^2}{2} - \frac{b}{2p}) = 7$; consequently $\frac{p}{2} + \sqrt{\frac{pp}{4} - s} = 3$, $\frac{p}{2} - \sqrt{\frac{pp}{4} - s} = 1$, $-\frac{p}{2} +$

$\sqrt{\frac{pp}{4}-q} = -2+\sqrt{-3}$, and $-\frac{p}{2}-\sqrt{\frac{pp}{4}-q} = -2-\sqrt{-3}$; which are the four roots of the equation $x^4-6x^2-16x+21$; to each of which let unity be added, and you will have 4, 2, $-1+\sqrt{-3}$, and $-1-\sqrt{-3}$, for the four roots of the equation proposed; whereof the two last are impossible.

And that these roots are truly assigned, may be easily proved by multiplying the equations, $y-4=0$, $y-2=0$, $y+1-\sqrt{-3}=0$, and $y+1+\sqrt{-3}=0$, thus arising, continually together; for, from thence, the very equation given will be produced.

The Resolution of Biquadratics by another Method.

In the method of *Des Cartes*, above explained, all biquadratic equations are supposed to be generated from the multiplication of two quadratic ones: but, according to the way which I am now going to lay down, every such equation is conceived to arise by taking the difference of two complete squares.

Here, the general equation $x^4+ax^3+bx^2+cx+d=0$ being proposed, we are to assume $\overline{x^2+\frac{1}{2}ax+A}\rceil^2-\overline{Bx+C}\rceil^2=x^4+ax^3+bx^2+cx+d$: in which A, B, and C represent unknown quantities, to be determined.

Then, $x^2+\frac{1}{2}ax+A$, and $Bx+C$ being actually involved, we shall have

$$\left.\begin{array}{llllll} x^4+ax^3 & +2Ax^2 & * & * \\ * \quad * & +\frac{1}{4}a^2x^2 & +aAx & +A^2 \\ * \quad * & -B^2x^2 & -2BCx & -C^2 \end{array}\right\} = x^4+ax^3+bx^2+$$

$cx+d$: from whence, by equating the homologous terms, will be given,

1. $2A+\frac{1}{4}a^2-B^2=b$, or $2A+\frac{1}{4}a^2-b=B^2$;
2. $aA-2BC=c$, or $aA-c=2BC$;
3. $A^2-C^2=d$, or $A^2-d=C^2$.

Let now the first and last of these equations be multiplied together, and the product will, evidently, be equal to $\frac{1}{4}$ of the square of the second, that is, $2\mathrm{A}^3 + \overline{\frac{1}{4}aa - b} \times \mathrm{A}^2 - 2d\mathrm{A} - d \times \overline{\frac{1}{4}aa - b}$ $(= \mathrm{B}^2\mathrm{C}^2) = \frac{1}{4} \times \overline{a^2\mathrm{A}^2 - 2ac\mathrm{A} + c^2}$ $(= \mathrm{B}^2\mathrm{C}^2)$. Whence, denoting the given quantities $\frac{1}{4}ac - d$, and $\frac{1}{4}c^2 + d, \times \overline{\frac{1}{4}aa - b}$ by k and l, respectively, there arises this cubic equation, $\mathrm{A}^3 - \frac{1}{2}b\mathrm{A}^2 + k\mathrm{A} - \frac{1}{2}l = 0$: by means whereof the value of A may be determined (as hath been already taught); from which, and the preceding equations, both B and C will be known, B being given from thence $= \sqrt{2\mathrm{A} + \frac{1}{4}aa - b}$, and $\mathrm{C} = \dfrac{a\mathrm{A} - c}{2\mathrm{B}}$.

The several values of A, B, and C, being thus found, that of x will be readily obtained: for $\overline{x^2 + \frac{1}{2}ax + \mathrm{A}}|^2 - \overline{\mathrm{B}x + \mathrm{C}}|^2$ being universally, in all circumstances of x, equal to $x^4 + ax^3 + bx^2 + cx + d$, it is evident, that when the value of x is taken such, that the latter of these expressions becomes equal to nothing, the former must likewise be $= 0$; and consequently $\overline{x^2 + \frac{1}{2}ax + \mathrm{A}}|^2 = \overline{\mathrm{B}x + \mathrm{C}}|^2$: whence by extracting the square root on both sides, $x^2 + \frac{1}{2}ax + \mathrm{A} = \pm \mathrm{B}x \pm \mathrm{C}$; which, solved, gives $x = \pm \frac{1}{2}\mathrm{B} - \frac{1}{4}a \pm \sqrt{\overline{\frac{1}{4}a - \frac{1}{2}\mathrm{B}}|^2 \pm \mathrm{C} - \mathrm{A}}$ $= \pm \frac{1}{2}\mathrm{B} - \frac{1}{4}a \pm \sqrt{\frac{1}{16}a^2 \mp \frac{1}{4}a\mathrm{B} + \frac{1}{4}\mathrm{B}^2 \pm \mathrm{C} - \mathrm{A}}$; exhibiting all the four different roots of the given equation, according to the variation of the signs.

This method will be found to have some advantages over that explained above. In the first place, there is no necessity here of being at the trouble of exterminating the second term of the equation, in order to prepare it for a solution: secondly, the equation $\mathrm{A}^3 - \frac{1}{2}b\mathrm{A}^2 + k\mathrm{A} - \frac{1}{2}l = 0$, here brought out, is of a more simple kind than that derived by the former method: and, thirdly (which advantage is the most considerable), the value of A, in this equation, will be *commensurate* and *rational* (and therefore the easier to be discovered), not only when all the roots of the given equation are *commensurate*, but when they are *irrational* and even *impossible;* as will appear from the examples subjoined.

Exam. 1. *Let there be given the equation* $x^4 + 12x - 17 = 0$.

Which being compared with the general equation $x^4 + ax^3 + bx^2 + cx + d = 0$, we have $a = 0$, $b = 0$, $c = 12$, and $d = -17$: therefore $k\ (\frac{1}{4}ac - d) = 17$, $l\ (\frac{1}{4}c^2 + d \times \overline{\frac{1}{4}aa - b}) = 36$; and consequently $A^3 - \frac{1}{2}bA^2 + kA - \frac{1}{2}l = A^3 + 17A - 18 = 0$; where it is evident, by bare inspection, that $A = 1$. Hence $B\ (= \sqrt{2A + \frac{1}{4}aa - b}) = \sqrt{2}$, $C\ (= \frac{aA - c}{2B}) = \frac{-12}{2\sqrt{2}} = -3\sqrt{2}$; and $x = \pm\frac{1}{2}\sqrt{2} \pm \sqrt{\frac{1}{2} \mp 3\sqrt{2} - 1} = \pm\frac{1}{2}\sqrt{2} \mp \sqrt{\mp 3\sqrt{2} - \frac{1}{2}}$. Therefore the four roots of the equation are $\frac{1}{2}\sqrt{2} + \sqrt{-3\sqrt{2} - \frac{1}{2}}$, $\frac{1}{2}\sqrt{2} - \sqrt{-3\sqrt{2} - \frac{1}{2}}$, $-\frac{1}{2}\sqrt{2} + \sqrt{3\sqrt{2} - \frac{1}{2}}$, and $-\frac{1}{2}\sqrt{2} - \sqrt{3\sqrt{2} - \frac{1}{2}}$; whereof the first and second are impossible.

Exam. 2. *Let the equation given be* $x^4 - 6x^3 - 58x^2 - 114x - 11 = 0$.

Here $a = -6$, $b = -58$, $c = -114$, and $d = -11$; whence $k\ (\frac{1}{4}ac - d) = 182$, $l\ (\frac{1}{4}cc + d \times \overline{\frac{1}{4}aa - b}) = 2512$; and therefore $A^3 + 29A^2 + 182A - 1256 = 0$. Where, trying the divisors 1, 2, 4, 157, &c. of the last term (according to the method delivered on p. 134) the third is found to succeed; the value of A being, therefore, $= 4$. Whence there is given $B = \sqrt{75} = 5\sqrt{3}$, $C = \frac{90}{10\sqrt{3}} = 3\sqrt{3}$, and $x\ (= \pm\frac{1}{2}B - \frac{1}{4}a \pm$

$\overline{\sqrt{\tfrac{1}{16}a^2 \mp \tfrac{1}{2}aB + \tfrac{1}{4}B^2 \pm C - A}}) = \pm \tfrac{5}{2}\sqrt{3} + \tfrac{3}{2} \pm \sqrt{17 \pm \tfrac{21}{2}\sqrt{3}}.$

Exam. 3. Let there be now proposed the literal equation $z^4 + 2az^3 - 37a^2z^2 - 38a^3z + a^4 = 0$.

This equation, by dividing the whole by a^4, and writing $x = \dfrac{z}{a}$, is reduced to the following numeral one, $x^4 + 2x^3 - 37x^2 - 38x + 1 = 0$. If, therefore, a, b, c, and d be now expounded by 2, -37, -38, and 1, respectively, we shall here have k $(\tfrac{1}{4}ac - d) = -20$, l $(\tfrac{1}{4}c^2 + d \times \overline{\tfrac{1}{4}aa - b}) = 399$; and, therefore, by substituting these values,

$$A^3 + \tfrac{37}{2}A^2 - 20A - \tfrac{399}{2} = 0.$$
$$\text{or, } 2A^3 + 37A^2 - 40A - 399 = 0.$$

Which equation, by the preceding methods, will be found to have three commensurable roots, $\tfrac{7}{2}$, -3, and -19: and any one of these may be used, the result, take which you will, coming out exactly the same. Thus, by taking -3, for A, we shall have $x^2 + x - 3 = \pm\sqrt{2} \times \overline{4x + 2}$: but, if A be taken $= \tfrac{7}{2}$, then will $x^2 + x + \tfrac{7}{2} = \pm\sqrt{5} \times \overline{3x + \tfrac{3}{2}}$: lastly, if A be taken $= -19$, then $x^2 + x - 19 = +6\sqrt{10}$. All which are, in effect, but one and the same equation, as will readily appear by squaring both sides of each, and properly transposing; whence the given equation $x^4 + 2x^3 - 37x^2 - 38x + 1 = 0$, will, in every case, emerge. And the same observation extends to all other cases, where there are more roots than one, it being indifferent which value we use; unless that some are to be preferred, as being the most simple and commodious.

Having given the general solution of biquadratic equations by the means of cubic ones, I shall now point out two or three particular cases, where every thing may be performed by the resolution of a quadratic only.

These are discovered from the preceding equations,

$$2A + \tfrac{1}{4}a^2 - b = B^2$$
$$aA - c = 2BC$$
$$\text{and } A^2 - d = C^2:$$

wherein, *if* A *be supposed* = 0, it is plain that $\frac{1}{4}a^2 - b = B^2$, $-c = 2BC$, and $-d = C^2$: whence $B = \sqrt{\frac{1}{4}aa - b}$, $C = \frac{-c}{2\sqrt{\frac{1}{4}aa - b}} = \sqrt{-d}$, and consequently $d = \frac{cc}{4f}$; by making $f = b - \frac{1}{4}aa$.

Therefore, in this case (wherein $d = \frac{cc}{4f}$), the general equation $x^2 + \frac{1}{2}ax + A = \pm Bx \pm C$, will become $x^2 + \frac{1}{2}ax = \pm x\sqrt{-f} \mp \sqrt{-d}$.

But, *if* B *be supposed* = 0; then will $2A + \frac{1}{4}a^2 - b = 0$, and also $aA - c = 0$; whence $A = \frac{1}{2}b - \frac{1}{8}a^2 = \frac{1}{2}f = \frac{c}{a}$; and therefore $C\ (= \sqrt{A^2 - d}) = \sqrt{\frac{1}{4}ff - d}$: so that in this case (where $c = \frac{af}{2}$) the general equation becomes $x^2 + \frac{1}{2}ax + \frac{1}{2}f = \pm\sqrt{\frac{1}{4}ff - d}$; which, solved, gives $x = -\frac{1}{4}a \pm \sqrt{\overline{\tfrac{1}{4}a}\,\rceil^2 - \frac{1}{2}f \pm \sqrt{\frac{1}{4}ff - d}}$.

Lastly, *if* C *be supposed* = 0, then will $aA - c = 0$, and $A^2 - d = 0$; consequently $A = \frac{c}{a} = \sqrt{d}$, and $B\ (= \sqrt{2A + \frac{1}{4}a^2 - b}) = \sqrt{\frac{2c}{a} - f}$: therefore, in this case (where $d = \frac{cc}{aa}$), we shall have $x^2 + \frac{1}{2}ax + \frac{c}{a} = \pm x\sqrt{\frac{2c}{a} - f}$.

From the whole of which it appears, that if c be $= \frac{af}{2}$, or d, either, equal to $\frac{cc}{4f}$, or to $\frac{cc}{aa}$ (f being =

$b - \frac{1}{4}aa$); then, the roots of the given equation, $x^4 + ax^3 + bx^2 + cx + d = 0$, may be obtained by the resolution of a quadratic, only.

Exam. 1. *Let there be given* $x^4 * - 25x^2 + 60x - 36 = 0$.

Here $a = 0$, $b = -25$, $c = 60$, and $d = -36$; therefore, f $(= -25)$ being $= \frac{cc}{4d}$ $(= -25)$, we have (*by case* 1) $x^2 + \frac{1}{2}ax = \pm x\sqrt{-f} \mp \sqrt{-d}$; that is, $x^2 = \pm 5x \mp 6$: which, solved, gives $x = \pm \frac{5}{2} \pm \sqrt{\frac{25}{4} \mp 6}$, that is, $x = \frac{5}{2} \pm \frac{1}{2}$, or $x = -\frac{5}{2} \pm \frac{7}{2}$: so that 3, 2, 1, and -6 are the four roots of the equation propounded.

Exam. 2. *Let there be now given* $x^4 + 2qx^3 + 3q^2x^2 + 2q^3x - r^4 = 0$.

Then, a being $= 2q$, $b = 3q^2$, $c = 2q^3$, and $d = -r^4$, thence will f $(= b - \frac{1}{4}aa) = 2q^2$, and $\frac{af}{2}$ $(= 2q^3) = c$; and so (the example belonging to *case* 2) we have x $(= -\frac{1}{2}a + \sqrt{\overline{\frac{1}{2}a}|^2 - \frac{1}{2}f \pm \sqrt{\frac{1}{4}ff - d}}) = -\frac{1}{2}q \pm \sqrt{-\frac{3}{4}qq \pm \sqrt{q^4 + r^4}}$.

Exam. 3. *Lastly, suppose there were given the equation* $x^4 - 9x^3 + 15x^2 - 27x + 9 = 0$.

Here, a being $= -9$, $b = 15$, $c = -27$, and $d = 9$, it is evident that $\frac{cc}{a}$ $(= 9) = d$ $(= 9)$: therefore, *by case* 3, we have $x^2 + \frac{1}{2}ax + \frac{c}{a} = \pm x\sqrt{\frac{2c}{a} + \frac{1}{4}aa - b}$;

that is, $x^2 - 4\frac{1}{2}x + 3 \left(= \pm x\sqrt{6 + \frac{81}{4} - 15}\right) = \pm \frac{3}{2}x\sqrt{5}$: which, solved, gives

$$x = \frac{9 \pm 3\sqrt{5} \pm \sqrt{78 \pm 54\sqrt{5}}}{4}.$$

The Resolution of Literal Equations, wherein the given and the unknown quantity are alike affected.

Equations of this kind, in which the given and the unknown quantities can be substituted, alternately, for each other, without producing a new equation, are always capable of being reduced to others of lower dimensions. *In order to such a reduction, let the equation, if it be of an even dimension, be first divided by the equal powers of its two quantities in the middle term; then assume a new equation, by putting some quantity (or letter) equal to the sum of the two quotients that arise by dividing those quantities one by the other, alternately; by means of which equation, let the said quantities be exterminated; whence a numeral equation will emerge, of half the dimensions with the given literal one.*

But, if the equation propounded be of an odd dimension, let it be, first, divided by the sum of its two quantities, so will it become of an even dimension, and its resolution will therefore depend upon the preceding rule.

Exam. 1. *Let there be given the equation* $x^4 - 4ax^3 + 5a^2x^2 - 4a^3x + a^4 = 0$.

Here, dividing by a^2x^2, we have $\frac{xx}{aa} - \frac{4x}{a} + 5 - \frac{4a}{x} + \frac{aa}{xx} = 0$ (or $\frac{xx}{aa} + \frac{aa}{xx} - 4 \times \frac{x}{a} + \frac{a}{x} + 5 = 0$, by joining the corresponding terms); and, by making $z = \frac{x}{a} + \frac{a}{x}$, and squaring both sides, we have also, $z^2 = \frac{xx}{aa} + 2 + \frac{aa}{xx}$, or $z^2 - 2 = \frac{xx}{aa} + \frac{aa}{xx}$. Therefore, by substituting these values, our equation becomes $z^2 - 2 - 4z + 5 = 0$, or $z^2 - 4z = -3$; whence $z = 3$. But $\frac{x}{a} + \frac{a}{x}$ being $= z$, we have $x^2 - zax = -a^2$; and consequently $x = \frac{1}{2}za \pm \sqrt{\frac{1}{4}a^2z^2 - aa} = \frac{1}{2}a \times \overline{z \pm \sqrt{zz - 4}} = \frac{1}{2}a \times \overline{3 \pm \sqrt{5}}$, *in the present case.*

Exam. 2. *Let there be given* $x^5 + 4ax^4 - 12a^2x^3 - 12a^3x^2 + 4a^4x + a^5 = 0$.

In this case, we must first divide by $x + a$, and the quotient will come out $x^4 + 3ax^3 - 15a^2x^2 + 3a^3x + a^4 = 0$: whence, by proceeding as in the former example, we have $\frac{xx}{aa} + \frac{aa}{xx} + 3 \times \overline{\frac{x}{a} + \frac{a}{x}} - 15 = 0$, or $z^2 - 2 + 3z - 15 = 0$, and from thence $z = \frac{\sqrt{77} - 3}{2}$.

Exam. 3. *Suppose there were given* $7x^6 - 26ax^5 - 26a^5x + 7a^6 = 0$.

This, divided by a^3x^3, becomes $7 \times \overline{\frac{x^3}{a^3} + \frac{x^3}{x^3}} - 26 \times \overline{\frac{x^2}{a^3} + \frac{a^2}{x^2}} = 0$. Now, making, as before, $z = \frac{x}{a} + \frac{a}{x}$, we have $z^2 - 2 = \frac{x^2}{a^2} + \frac{a^2}{x^2}$; and, multiplying again by $z = \frac{x}{a} + \frac{a}{x}$, we likewise have $z^3 - 2z = \frac{x^3}{a^3} + \frac{a}{x} + \frac{x}{a} + \frac{a^3}{x^3} = \frac{x^3}{a^3} + z + \frac{a^3}{x^3}$; and, therefore, $z^3 - 3z = \frac{x^3}{a^3} + \frac{a^3}{x^3}$: which values being substituted as above, our equation becomes $7 \times \overline{z^3 - 3z} - 26 \times \overline{z^2 - 2} = 0$, or $7z^3 - 26z^2 - 21z + 52 = 0$.

Where, trying the divisors of the last term, which are 1, 2, 4, 13, &c. the third is found to answer; z, consequently, being $= 4$.

Exam. 4. *Wherein let there be given* $2x^7 - 13a^2x^5 - 13a^5x^2 + 2a^7 = 0$.

Here, dividing, first, by $x + a$, the quotient will be $2x^6 - 2ax^5 - 11a^2x^4 + 11a^3x^3 - 11a^4x^2 - 2a^5x + 2a^6 = 0$; which, divided again by a^3x^3, gives $2 \times \overline{\frac{x^3}{a^3} + \frac{a^3}{x^3}} - 2 \times \overline{\frac{x^2}{a^2} + \frac{a^2}{x^2}} - 11 \times \overline{\frac{x}{a} + \frac{a}{x}} + 11 = 0$, that is, $2 \times \overline{z^3 - 3z} - 2 \times \overline{z^2 - 2} - 11z + 11 = 0$, or $2z^3 - 2z^2 - 17z + 15 = 0$ (*vid. p.* 119): whence $z = 3$.

A *literal equation* may be made to correspond with a *numeral one*, by substituting a unit in the room of the given quantity (or letter): and equations that do not seem, at first, to belong to the preceding class may sometimes be reduced to such, by a proper substitution; that is, by putting the quotient of the first term divided by the last, equal to some new unknown quantity (or letter) raised to the power expressing the dimension of the equation. Thus, if the equation given be $2x^4 + 24x^3 - 315x^2 + 216x + 162 = 0$; by putting $\frac{2x^4}{162} = y^4$, we have $x = 3y$: whence, after substitution, the given equation becomes $162y^4 + 648y^3 - 2835y^2 + 648y + 162 = 0$: which now answers to the rule, and may be reduced down to $2y^4 + 8y^3 - 35y^2 + 8y + 2 = 0$.

Of the Resolution of Equations by Approximation and Converging Series.

The methods hitherto given, for finding the roots of equations, are either very troublesome and laborious, or else confined to particular cases; but *that* by converging series, which we are here going to explain, is universal, extending to all kinds of equations; and, though not accurately true, gives the value sought, with little trouble, to a very great degree of exactness. When an equation is proposed to be solved by this method, the root thereof must, first of all, be nearly estimated (which, from the nature of the problem, and a few trials, may, in most cases, be very easily done); and some letter, or unknown quantity (as z) must be assumed, to express the difference between that value, which we will call r, and the true value (x); then, instead of x, in the given equation, you are to substitute its equal, $r \pm z$, and there will emerge a new equation, affected only with z and known quantities; wherein all the terms having two or more dimensions of z, may be rejected, as inconsiderable in respect of the rest; which being done, the value of z will be found, by the resolution of a simple equation; from whence that of x $(= r \pm z)$ will also be known. But, if this value should not be thought sufficiently near the truth, the operation may be repeated, by substituting the said value instead of r, in the equation exhibiting the value of z; which will give a second correction for the value of x.

As an example hereof, let the equation $x^3 + 10x^2 + 50x = 2600$, be proposed: thence, since it appears that x must, in this case, be somewhat greater than 10, let r be put $= 10$, and $r + z = x$; which value being substituted for x, in the given equation, we have $r^3 + 3r^2z + 3rz^2 + z^3 + 10r^2 + 20rz + 10z^2 + 50r + 50z = 2600$: this, by rejecting all the terms wherein two or more dimensions of z are concerned, is reduced to $r^3 + 3r^2z + 10r^2 + 20rz + 50r + 50z = 2600$; whence z comes out $= \dfrac{2600 - r^3 - 10r^2 - 50r}{3r^2 + 20r + 50}$ $= 0.18$, nearly: which, added to 10 ($= r$), gives 10,18 for the value of x. But, in order to repeat the operation, let this value be substituted for r, in the last equation, and you will have $z = -,0005347$; which, added to 10,18, gives 10,1794653, for the value of x, a second time corrected. And, if this last value be again substituted for r, you will have a third correction of x; from whence a fourth may, in like manner, be found; and whence, by repeating the operation, the next value of x will be found $= 2,1544$.

The manner of approximating hitherto explained, as all the powers of the converging quantity after the first are rejected, only doubles the number of figures at every operation. But I shall now give the investigation of other rules, or *formulæ*, whereby the number of places may be tripled, quadrupled, and even quintupled, at every operation.

Let there be assumed the general equation $az + bz^2 + cz^3 + dz^4$, &c. $= p$; z, as above, being the converging quantity, and a, b, c, d, &c. such known numbers as arise by substituting in the original equation, after the value of the required root is nearly estimated.

Then, by transposition and division, we shall have $z = \frac{p}{a} - \frac{bz^2}{a} - \frac{cz^3}{a} - \frac{dz^4}{a}$, &c. from whence, by rejecting all the terms after the first, and writing $q = \frac{p}{a}$, there will be give $z = q$: which value, taking in only one term of the given series, I call an approximation of the first degree, or order.

To obtain an approximation of the second degree, or such a one as shall include two terms of the series, let the value of z, found as above, be now substituted in the second term $\frac{bz^2}{a}$, rejecting all the following ones; so shall $z = \frac{p}{a} - \frac{bq^2}{a} = q - \frac{bq^2}{a}$, which triples the number of figures at every operation.

For an approximation of the third degree, let this last value of z be now substituted in the second and third terms, neglecting every where all such quantities as have more than three dimensions of q: whence z will be had $\left(= q - \frac{bq^2}{a} + \frac{2b^2q^3}{aa} - \frac{cq^3}{a}\right) = q - \frac{b}{a}q^2 + \frac{2bb - ac}{aa}q^3$.

The manner of continuing these approximations is sufficiently evident: but there are others, of the same degrees, differing in form, which are rather more commodious, and whereof the investigation is also somewhat different.

It is evident from the given equation, that $z = \frac{p}{a + bz + cz^2 + dz^3, \&c.}$. If, therefore, the first value of z, found as above, be substituted in the denominator, and all the terms after the second be rejected, we shall have $z = \frac{p}{a+bq} = \frac{ap}{aa+bp}$; which is an approximation of the second degree.

But, if for z you write its second value, $q - \frac{bq^2}{a}$, you will then have z $\left(= \frac{p}{a + bq - \frac{b^2q^2}{a} + cq^2}\right) = \frac{p}{a + bq - \overline{\frac{bb}{a} - c} \cdot q^2}$; being an approximation of the third degree.

Again, by writing $q - \frac{b}{a} q^2 + \frac{2bb - ac}{aa} \cdot q^3$ in the room of z, and neglecting every where all such terms as have more than three dimensions of q, you will have

$$z \left(= \frac{p}{a + bq - \frac{bb}{a} q^2 + \frac{2b^3 - abc}{aa} \cdot q^3 + c \times \overline{q^2 - \frac{2b}{a} \cdot q^3} + dq^3}\right)$$

$$= \frac{p}{a + bq - \overline{\frac{bb}{a} - c} \cdot q^2 + \overline{\frac{2b^3}{aa} - \frac{3bc}{a} + d} \cdot q^3}$$: which is an approximation of the fourth degree.

It is observable, that the powers of the converging quantity q, in the former approximations, stand, all of them, in the numerator; but *here*, in the denominator;

but there is an artifice for bringing them alike, into both, and thereby lessening the number of dimensions, without taking away from the the rate of convergency.

To begin with the approximation $z = \frac{p}{a + bq - \overline{\frac{bb}{a} - c} \cdot q^2}$, which is of the third degree, put $s = \frac{b}{a} - \frac{c}{b}$ = the coefficient of the last term of the denominator divided by that of the last but one; so shall $z = \frac{p}{a + bq - bsq^2}$; whereof the numerator and the denominator, being equally multiplied by $1 + sq$, it becomes $z = \frac{p \times \overline{1 + sq}}{a + bq - bsq^2 + asq + bsq^2 - bs^2q^3}$: but, the approximation being only of the third degree, bs^2q^3 may be rejected, and so we have

$$z = \frac{p + pqs}{a + \overline{b + as} \cdot q} = \frac{\overline{a + sp} \cdot p}{aa + \overline{b + as} \cdot p}.$$

In the same manner, in order to exterminate the third dimension of q out of the equation,

$$z = \frac{p}{a + bq - \overline{\frac{bb}{a} - c} \cdot q^2 + \overline{\frac{2b^3}{aa} - \frac{3bc}{a} + d} \cdot q^3}$$

put $w = \frac{2b}{a} + \frac{ad - bc}{bb - ac}$ = the coefficient of the last term of the denominator divided by that of the last but one; then will $z = \frac{p}{a + bq - \overline{\frac{bb}{a} - c} \cdot q^2 + \overline{\frac{bb}{a} - c} \cdot wq^3}$ $= \frac{p}{a + bq - bsq^2 + bswq^3}$ (because $s = \frac{b}{a} - \frac{c}{b}$); whereof the terms being equally multiplied by $1 + wq$, &c. we thence have $z = \frac{p \times \overline{1 + wq}}{a + bq - bsq^2 + awq + bwq^2}$.

$$\doteq \frac{p\times\overline{1+wq}.}{a+\overline{b+aw}\,.\,q+\overline{w-s}\,.\,bq^2} = \frac{ap\times\overline{a+wp}}{a\times\overline{aa+\overline{b+aw}\,.\,p+\overline{w-s}\,.\,pp}}$$

: which is an approximation of the fourth degree, and quintuples the number of figures at every operation.

By pursuing the same method, other equations might be determined, to include five or more terms of the given series; but, then, they would be found more tedious, and perplexed in proportion; so that no real advantage, in practice, could be reaped therefrom. I shall, therefore, proceed now to illustrate what is laid down above by a few examples.

Exam. 1. *Let the equation given be* $x^2+20x=100$.

Here, x appearing, by inspection, to be something greater than 4, make $4+z=x$: then the given equation, by substitution, becomes $28z+z^2=4$. Therefore, in this case, $a=28$, $b=1$, $c=0$, &c. and $p=4$; and consequently $\frac{ap}{aa+bp}$ $(=\frac{112}{788}=\frac{28}{197})=$ 0.14213; which is one approximation of the value of z. But, if greater exactness be required, then s $(\frac{b}{a}-\frac{c}{b})$ being here $\frac{1}{28}$, and w $(\frac{2b}{a}+\frac{ad-bc}{bb-ac})=\frac{1}{14}$, we shall, according to our two last *formulæ*, have

$$z\ (=\frac{\overline{a+sp}\,.\,p}{aa+\overline{b+ds}\,.\,p}) = \frac{\overline{28+\frac{1}{7}}\times4}{28\times28+2\times4} = \frac{28+\frac{1}{7}}{28\times7+2} = \frac{197}{1386} = 0{,}14213564,$$

nearly; and

$$z\ (=\frac{ap\times\overline{a+wp}}{a\times\overline{aa+\overline{b+aw}\,.\,p+\overline{w-s}\,.\,pp}} = \frac{28\times4\times\overline{28+\frac{2}{7}}}{28\times\overline{784+12+\frac{4}{7}}} = \frac{28\times\overline{28+\frac{2}{7}}}{7\times\overline{796+\frac{1}{7}}} = \frac{28\times198}{49\times796+1} = \frac{5544}{39005} =$$

0.1421356236, more nearly; which value is true to the last figure.

Exam. 2. *Suppose the given equation, when prepared for a solution, to be* $768z + 48z^2 + z^3 = -96$.

In this case, $a = 768$, $b = 48$, $c = 1$, $d = 0$, $p = -96$, $q \left(= \frac{p}{a}\right) = -\frac{1}{8}$, $s \left(= \frac{b}{a} - \frac{c}{b}\right) = \frac{1}{16} - \frac{1}{48} = \frac{1}{24}$, and $w \left(= \frac{2b}{a}\right) + \frac{ad - bc}{bb - ac} = \frac{1}{8} - \frac{48}{48 \times 48 - 768} = \frac{1}{8} - \frac{1}{48 - 16} = \frac{3}{32}$. Therefore $z = \frac{p + pqs}{a + \overline{b + as} \cdot q} = \frac{-96 - 96 \times -\frac{1}{8} \times \frac{1}{24}}{768 + 48 + 32 \times -\frac{1}{8}} = \frac{-96 + \frac{1}{2}}{768 - 6 - 4} = \frac{-191}{1516} = -0.1259894$, nearly; or $z = \frac{p + pqw}{a + \overline{b + aw} \cdot q + \overline{w - s} \cdot bqq} = \frac{-96 - 96 \times -\frac{1}{8} \times \frac{3}{32}}{768 + 48 + 72 \times -\frac{1}{8} + \frac{5}{96} \times \frac{48}{64}} = \frac{-96 + \frac{9}{8}}{768 - 6 - 9 + \frac{5}{128}} = \frac{-96 \times 128 + 9 \times 16}{753 \times 128 + 5} = -\frac{12144}{96389} = -0{,}1259894802$, more nearly.

In the same manner the roots of other equations may be approached: but, to avoid trouble in preparing the equation for a solution, you may every where neglect all such powers of the converging quantity, z, as would rise higher than the degree or order of the approximation you intend to work by. And, further to facilitate the labour of such a transformation, the following general equations of the values of p, a, b, c, d, &c. may be used.

$p = k - \alpha r - \beta r^2 - \gamma r^3 - \delta r^4$, &c.

$a = \alpha + 2\beta r + 3\gamma r^2 + 4\delta r^3$, &c.

$b = \beta + 3\gamma r + 6\delta r^2 + 10\epsilon r^3$, &c.

$c = \gamma + 4\delta r + 10\epsilon r^2 +$ &c.

$d = \delta + 5\epsilon r +$ &c.

The original equation being $\alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \epsilon x^5$, &c. $= k$: from whence, by making $r + z = x$, the above values are deduced.

The better to illustrate the use of what is here laid down, I shall subjoin another example; wherein let there be given $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x$ (or $5x + 4x^2 + 3x^3 + 2x^4 + x^5$) $= 54321$; to find x, by an approximation of the second degree.

In this case, k being $= 54321$, $\alpha = 5$, $\beta = 4$, $\gamma = 3$, $\delta = 2$, and $\varepsilon = 1$, we have

$p = 54321 - 5r - 4r^2 - 3r^3 - 2r^4 - r^5$,
$a = 5 + 8r + 9r^2 + 8r^3 + 5r^4$, and
$b = 4 + 9r + 12r^2 + 10r^3$.

Which values, by assuming $r = 8$, will become $p = 11529$, $a = 25221$, and $b = 5964$: whence q $(= \frac{p}{a}) = 0{,}45$, and z $(= \frac{p}{a + bq}) = \frac{11529}{25221 + 2683} = 0{,}41$; and therefore x $(= r + z) = 8{,}41$, nearly.

To repeat the operation, let 8,41 be now substituted for r; so shall $p = 135{,}92$, $a = 30479$, $b = 6876$, q $(= \frac{p}{a}) = 0.00445$, and z $(= \frac{p}{a + bq}) = \frac{135{,}92}{30479 + 30} = 0{,}004455$: which, added to 8,41, gives 8,414455, for the next value of x.

The *formulæ*, or approximations determined in the preceding pages, are general, answering [illegible] equations of all degrees howsoever affected; but [illegible] the extraction of the roots of *pure* powers, the process [illegible] be [illegible]te simple, and the theorems themselves very muc[illegible]viated.

For let $x^n = k$ be the equation whereof the root x is to be extracted; then, by assuming r nearly equal to x, and making $r \times \overline{1 + z} = x$, our equation will become $r^n \times \overline{1 + z}\,]^n = k$, or $\overline{1 + z}\,]^n = \frac{k}{r^n}$, that is, $1 + nz + n \cdot \frac{n-1}{2} z^2 + n \cdot \frac{n-1}{2} \cdot$ [illegible] $+ n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot z^4$, &c. $= \frac{k}{r^n}$: from whence, by transposition and division, $z + \frac{n-1}{2} \cdot z^2 + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot z^3$

$$+\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdot z^4, \text{ \&c.} = \frac{k-r^n}{nr^n}.$$

Here, by a comparison with the general equation, $az + bz^2 + cz^3 + dz^4$, &c. $= p$, we have $a = 1$, $b = \frac{n-1}{2}$, $c = \frac{n-1}{2}\cdot\frac{n-2}{3}$, $d = \frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}$, &c. and $p = \frac{k-r^n}{nr^n}$: whence $q\ (\frac{p}{a}) = p$; $s\ (\frac{b}{a} - \frac{c}{b}) = \frac{n-1}{2} - \frac{n-2}{3} = \frac{n+1}{6}$; and $w\ (\frac{2b}{a}$

$$+\frac{\frac{ad}{b}-c}{b-\frac{ac}{b}}) = \frac{n-1}{1} + \frac{\frac{1}{12}\cdot\overline{n-2}\cdot\overline{n-3} - \frac{1}{6}\cdot\overline{n-1}\cdot\overline{n-2}}{{}^{1}\cdot\overline{n-1} - \frac{1}{3}\cdot\overline{n-2}}$$

$$= \frac{n-1}{1} + \frac{\overline{n-2}\cdot\overline{n-3} - \overline{2n-2}\cdot\overline{n-2}}{2\cdot\overline{n+1}} = \frac{n-1}{1} + \frac{n-2}{2\cdot\overline{n+1}} \times \overline{n-3-2n+2} = \frac{n-1}{1} + \frac{n-2}{2\cdot\overline{n+1}} \times -\overline{n+1} = \frac{n-1}{1} - \frac{n-2}{2} = \frac{n}{2}.$$

Therefore, for an approximation of the third degree, we have

$$z = \frac{a + s\text{[illegible]}p}{aa + b + \text{[illegible]}\cdot p} \text{[illegible]} \frac{1 + \frac{1}{6}p\cdot\overline{n+1}\cdot p}{\overline{1 + \frac{n-1}{2} + \frac{n+1}{6}}\cdot p} = \frac{p + \overline{n+1}\cdot\frac{1}{6}p^2}{1 + \overline{2n-1}\cdot\frac{1}{3}p}$$

and for an approximation of the fourth degree,

$$z = \frac{p \times \overline{1 + wq}}{\overline{a + b + aw}\cdot q + \overline{w - s}\cdot bq^2} = \frac{\text{[illegible]}p + \frac{1}{2}np^2}{1 + \frac{n-1}{2} + \text{[illegible]} + \overline{\frac{n}{2} - \frac{n+1}{6}}\cdot\frac{n-1}{2}\cdot p^2}.$$

$= \frac{p + \frac{1}{2}np^2}{1 + \frac{2n-1}{2}\cdot p + \frac{2n-1}{2}\cdot\frac{n-1}{6}\cdot p^2}$. Hence it is evident that the root x $(r \times \overline{1+z})$ of the given equation $x^n = k$, will be equal to $r + \frac{rp \times \overline{1 + \overline{n+1}\cdot\frac{1}{6}p}}{1 + \overline{2n-1}\cdot\frac{1}{3}p}$, *nearly;* and equal to $r + \frac{rp \times \overline{1 + \frac{1}{2}np}}{1 + \frac{2n-1}{2}\cdot p + \frac{\overline{2n-1}\cdot\overline{n-1}\cdot p^2}{12}}$, *more nearly.*

But both these theorems will be rendered a little more commodious, by putting $v = \frac{nr^n}{k - r^n}$, and substituting $\frac{1}{v}$, in the place of its equal, p, whence, after proper reduction, x will be had $= r + \frac{r \times \overline{6v + n + 1}}{v \times \overline{6v + 4n - 2}}$, *nearly;* and equal to $r + \frac{r \times \overline{2v + n}}{v \times \overline{2v + \overline{2n-1} + \frac{1}{6}\cdot\overline{n-1}\cdot\overline{2n-1}}}$, *more nearly.*

I shall now put down an example or two, to show the use and great exactness of these last expressions.

1. Let the equation given be $x^2 = 2$, or, which is the same thing, let the square root of 2 be required.

Then, assuming $r = 1.4$, we have $n = 2$, $k = 2$, $v\ \left(\frac{nr^n}{k - r^n} = \frac{2 \times 1{,}96}{2 - 1.96}\right) = 98$; and therefore $r + \frac{r \times \overline{6v + n + 1}}{v \times \overline{6v + 4n - 2}} = 1{,}4 + \frac{\text{[illegible]} \times 591}{9\text{[illegible]} \times 594} = 1{,}4 + \frac{197}{70 \times 198} = 1{,}4 + \frac{197}{13860} = 1{,}41421356$; which is the value of x according to the former approximation;

but, according to the latter, the answer will come out $1.4 + \frac{5544}{39005} = 1.41421356236$; which is true to the last figure; and, if with this number the operation be repeated, you will have the answer true to nearly 60 places of decimals.

2. Let it be required to extract the cube root of 1728. Here, taking $r = 11$, we shall have $v\ (\frac{nr^n}{k - r^n}) = \frac{3993}{397} = 10.05793$; and therefore $r + \frac{r \times \overline{2v+n}}{\overline{2v+2n-1} \times v + \frac{1}{6} \times \overline{n-1} \times \overline{2n-1}} = 11{,}99998$; which differs from truth by only $\frac{1}{50000}$ part of a unit.

3. Let it be proposed to extract the cube root of 500. Here, the required root appearing to be less than 8, but nearer to 8 than 7, let r be taken $= 8$, and we shall have v ([illegible] $\frac{3 \times 512}{-12}) = -128$; and therefore $r + \frac{r \times \overline{2v+n}}{2v + \text{[illegible]} \times v + \frac{1}{6} \times \overline{n-1} \times \overline{2n-1}}$ $= 8 - \frac{6072}{96389} = $ 7,93[illegible]05259936; which number is true to the last place.

4. Lastly, let it be proposed to extract the first sursolid root of 125000. In which case, k being $= 125000$, $n = 5$, $r = 10$, and $v = 20$, the required root will be found $=$ 10,456389.

Besides the different approximations hitherto delivered, there are various other ways whereby the roots of equations may be approached; but, of these, none more general, and easy in practice, than the following:

Let the general equation, $az + bz^2 + cz^3 + dz^4 + ez^5$, &c. $= p$, be here resumed; which, by division, becomes $z = \dfrac{1}{\frac{a}{p} + \frac{b}{p}z + \frac{c}{p}z^2 + \frac{d}{p}z^3 + \frac{e}{p}z^4 + \text{\&c.}}$. If, therefore, we make $A = \dfrac{a}{p}$; and neglect all the terms after the first, we shall have $z = \dfrac{1}{A}$; being an approximation into of the first degree.

And if this value of z be now substituted in the second term, and all the following ones be rejected, we shall then have $z = \dfrac{1}{\frac{a}{p} + \frac{b}{p} \times \frac{1}{A}} = \dfrac{A}{\frac{a}{p}A + \frac{b}{p}} = \dfrac{A}{B}$ (by making $B = \dfrac{aA + b}{p}$); which is an approximation of the second degree.

In order now to get an approximation of the third degree, let this last value be substituted in the second term, neglecting all the terms after the third; so shall $z = \dfrac{1}{\frac{a}{p} + \frac{b}{p} \times \frac{A}{B} + \frac{c}{p}z^2}$: but here, in the room of z^2, either of the squares of the preceding values of z, or their rectangle may be substituted, that is, either $\dfrac{1}{A} \times \dfrac{1}{A}$, $\dfrac{A}{B} \times \dfrac{A}{B}$, or $\dfrac{1}{A} \times \dfrac{A}{B}$; but the last of these $(= \dfrac{1}{B})$ is the most commodious; whence we have $z = \dfrac{B}{\frac{a}{p}B + \frac{b}{p}A + \frac{c}{p}} = \dfrac{B}{C}$; supposing $C = \dfrac{aB + bA + c}{p}$.

Again, for an approximation of the fourth degree, we have $\dfrac{b}{p}z = \dfrac{b}{p} \times \dfrac{B}{C}$; $\dfrac{c}{p}z^2 = \dfrac{c}{p} \times \dfrac{B}{C} \times \dfrac{A}{B} = \dfrac{c}{p} \times \dfrac{A}{C}$; and $\dfrac{d}{p}z^3 = \dfrac{d}{p} \times \dfrac{B}{C} \times \dfrac{A}{B} \times \dfrac{1}{A} = \dfrac{d}{p} \times \dfrac{1}{C}$; which values being substituted in the general equation, and all the terms after the four first rejected, there now comes out $z = \frac{1}{\frac{a}{p} + \frac{bB}{pC} + \frac{cA}{pC} + \frac{d}{pC}} = \frac{C}{\frac{a}{p}C + \frac{b}{p}B + \frac{c}{p}A + \frac{d}{p}}$ $= \frac{C}{D}$; by making $D = \frac{aC + bB + cA + d}{p}$.

In like manner, for an approximation of the fifth degree, we shall have $\frac{b}{p}z = \frac{b}{p} \times \frac{C}{D}$, $\frac{c}{p}z^2 = \frac{c}{p} \times \frac{C}{D} \times \frac{B}{C}$ $= \frac{cB}{pD}$, $\frac{d}{p}z^3 = \frac{d}{p} \times \frac{C}{D} \times \frac{B}{C} \times \frac{A}{B} = \frac{dA}{pD}$, and $\frac{e}{p}z^4 =$ $\frac{C}{D} \times \frac{B}{C} \times \frac{A}{B} \times \frac{1}{A} = \frac{e}{pD}$; and consequently z $= \frac{D}{\frac{a}{p}D + \frac{b}{p}C + \frac{c}{p}B + \frac{d}{p}A + \frac{e}{p}} = \frac{D}{E}$; supposing $E = \frac{aD + bC + cB + dA + e}{p}$. Whence the law of continuation is manifest; whereby it appears, that, if there be taken $A = \frac{a}{p}$, $B = \frac{aA + b}{p}$, $C = \frac{aB + bA + c}{p}$, $D = \frac{aC + bB \text{ [illegible] }}{p}$, $E = \frac{aD + bC + cB + dA + e}{p}$, $F = \frac{aE + bD + cC + \text{[illegible]} + f}{p}$, $G = \frac{aF + bE + cD + dC + eB + fA + g}{p}$, &c. then will $\frac{1}{A}, \frac{A}{B}, \frac{B}{C}, \frac{C}{D}, \frac{D}{E}, \frac{E}{F}, \frac{F}{G}$, &c. be so many successive approximations to the value of z, ascending gradually from the lowest to the superior orders.

An example will help to explain the use of what is above delivered; wherein we will suppose the equation given to be $12z + 6z^2 + z^3 = 2$.

Here $a = 12$, $b = 6$, $c = 1$, $d = 0$, $e = 0$, &c. and $p = 2$; whence $A \left(= \frac{a}{p}\right) = 6$, $B \left(= \frac{aA + b}{p}\right) = \frac{12 \times 6 + 6}{2}$ $= 39$, $C \left(= \frac{aB + bA + c}{p}\right) = \frac{12 \times 39 + 6 \times 6 + 1}{2} =$

$\frac{505}{2}$, D $\left(= \frac{aC + bB + cA + d}{p}\right) = \frac{6 \times 505 + 6 \times 39 + 6}{2}$ $= 1635$, *&c.*

$$\text{Therefore, } \frac{A}{B} = \frac{2}{13} = z, \textit{ nearly.}$$

$$\frac{B}{C} = \frac{78}{505} = z, \textit{ more nearly.}$$

$$\frac{C}{D} = \frac{101}{654} = z, \textit{ still nearer.}$$

From the same equations the general values of B, C, D, *&c.* may be easily found, in known terms, independent of each other.

Thus B $\left(= \frac{aA}{p} + \frac{b}{p}\right) = \frac{a^2}{p^2} + \frac{b}{p}$ (because A $= \frac{a}{p}$); also C $\left(= \frac{aB}{p} + \frac{bA}{p} + \frac{c}{p}\right) = \frac{a^3}{p^3} + \frac{2ab}{p^2} + \frac{c}{p}$; and D $\left(= \frac{aC}{p} + \frac{bB}{p} + \frac{cA}{p} + \frac{d}{p}\right) = \frac{a^4}{p^4} + \frac{3a^2b}{p^3} + \frac{2ac + bb}{p^2}$ $+ \frac{d}{p}$, &c. Therefore

$$\frac{A}{B} = \frac{ap}{a^2 + bp};$$

$$\frac{B}{C} = \frac{p \times \overline{a^2 + bp}}{a^3 + 2abp + cp^2};$$

$$\frac{C}{D} = \frac{p \times \overline{a^3 + 2abp + cp^2}}{a^4 + 3a^2bp + \overline{2ac + bb} \cdot p^2 + dp^3};$$

$$\frac{D}{E} = \frac{p \times \overline{a^4 + 3a^2bp + \overline{2ac + bb} \cdot p^2 + dp^3}}{a^5 + 4a^3bp + \overline{3ac + 3bb} \cdot ap^2 + \overline{bc + ad} \cdot 2p^3 + ep^4},$$

&c. which are so many different approximations to the value of z.

Thus far regard has been had to equations which consist of the simple powers of one unknown quantity, and are no ways affected, either by surds or fractions. If either of these kinds of quantities be concerned in an equation, the usual way is to exterminate them by multiplication, or involution (as has been taught in sect. IX.) But as this method is, in many cases, very laborious, and in others altogether impracticable, especially where several surds are concerned in the same equation, it may not be amiss to show how the method of converging series may be also extended to these cases, without any such previous reduction. In order to which it will be necessary to premise, that if A + B represent a compound quantity, consisting of two terms, and the latter (B) be but small in comparison of the former; then will

$$\left.\begin{array}{l}
1^\circ.\ \frac{1}{A+B}=\frac{1}{A}-\frac{B}{A^2}\text{ or }\frac{1}{A}\times\overline{1-\frac{B}{A}}\\
2^\circ.\ \overline{A+B}|^{\frac{1}{2}}=A^{\frac{1}{2}}+\frac{B}{A2^{\frac{1}{2}}}\text{ or }A^{\frac{1}{2}}+\frac{A^{\frac{1}{2}}B}{2A}\\
3^\circ.\ \frac{1}{\overline{A+B}|^{\frac{1}{2}}}=\frac{1}{A^{\frac{1}{2}}}-\frac{B}{2A^{\frac{3}{2}}}\text{ or }\frac{1}{A^{\frac{1}{2}}}-\frac{B}{2A\times A^{\frac{1}{2}}}\\
4^\circ.\ \overline{A+B}|^{\frac{1}{3}}=A^{\frac{1}{3}}+\frac{B}{3A^{\frac{2}{3}}}\text{ or }A^{\frac{1}{3}}+\frac{A^{\frac{1}{3}}B}{3A}\\
5^\circ.\ \frac{1}{\overline{A+B}|^{\frac{1}{3}}}=\frac{1}{A^{\frac{1}{3}}}-\frac{B}{3A^{\frac{4}{3}}}\text{ or }\frac{1}{A^{\frac{1}{3}}}-\frac{B}{3A\times A^{\frac{1}{3}}}\\
6^\circ.\ \overline{A+B}|^{\frac{1}{4}}=A^{\frac{1}{4}}+\frac{B}{4A^{\frac{3}{4}}}\text{ or }A^{\frac{1}{4}}+\frac{BA^{\frac{1}{4}}}{4A}\\
7^\circ.\ \frac{1}{\overline{A+B}|^{\frac{1}{4}}}=\frac{1}{A^{\frac{1}{4}}}-\frac{B}{4A^{\frac{5}{4}}}\text{ or }\frac{1}{A^{\frac{1}{4}}}-\frac{B}{4A\times A^{\frac{1}{4}}}
\end{array}\right\}\text{, nearly.}$$

All which will appear evident from the general theorem at p. 41: from whence these particular equations, or theorems, may be continued at pleasure; the values here exhibited being nothing more than the two first terms of the series there given. But now, to apply them to the purpose above mentioned, let there be given $\sqrt{1+x^2}+\sqrt{2+x^2}+\sqrt{3+x^2}=10$, as an example, where, x being about 3, let $3+e$ be therefore substituted for x, rejecting all the powers of e above the first, as inconsiderable, and then the given equation will stand thus, $\sqrt{10 + 6e} + \sqrt{11 + 6e} + \sqrt{12 + 6e} = 10$: but, by *theorem* 2, $\sqrt{10 + 6e}$ will be $= \sqrt{10} + \frac{3\sqrt{10} \times e}{10}$, nearly; for, in this case, A = 10, and B = 6e, and therefore $A^{\frac{1}{2}} + \frac{A^{\frac{1}{2}}B}{2A} = \sqrt{10} + \frac{3\sqrt{10} \times e}{10}$: in like manner is $\sqrt{11 + 6e} = \sqrt{11} + \frac{3\sqrt{11} \times e}{11}$, &c. and consequently $\sqrt{10} + \frac{3\sqrt{10} \times e}{10} + \sqrt{11} + \frac{3\sqrt{11} \times e}{11} + \sqrt{12} + \frac{3\sqrt{12} \times e}{12} = 10$; which, contracted, gives $9.944 + 2.718e = 10$; whence $2.718e = .056$, and $e = .0205$; consequently $x = 3.0205$, nearly. Wherefore, to repeat the operation, let $3.0205 + e$ be now substituted for x; then will $\sqrt{10.12342 + 6.041e} + \sqrt{11.12342 + 6.041e} + \sqrt{12.12342 + 6.041e} = 10$; whence, by *theorem* 2, $\sqrt{10.12342} + \frac{6.041e}{2\sqrt{10.12342}} + \sqrt{11.12342} + \frac{6.041e}{2\sqrt{11.12342}} + \sqrt{12.12342} + \frac{6.041e}{2\sqrt{12.12342}} = 10$, or $9{,}9987814 + 2.7224e = 10$: from which e comes out $= .000447$, and therefore $x = 3.020947$; which is true to the last place.

Again, let it be proposed to find the root of the equation $\frac{20x}{\sqrt{16 + 5x + x^2}} + \frac{\sqrt{5x + x^2}}{25} = 34$. Put $20 + e = x$; then, by proceeding as before, we shall have $\frac{400 + 20e}{\sqrt{516 + 45e}} + \frac{20 + e \times \sqrt{405 + 40e}}{25} = 34$: but (*by theorem* 3) $\frac{1}{\sqrt{516 + 45e}}$ is nearly $= \frac{1}{\sqrt{516}} -$

$\frac{45e}{1032 \times \sqrt{516}}$, and (*by theorem* 2) $\sqrt{405 + 40e}$ [illegible] $\sqrt{405} + \frac{20e}{\sqrt{405}}$: which values being substituted above, our equation becomes

$$\overline{400+20e} \times \overline{\frac{1}{\sqrt{516}} - \frac{45e}{1032 \times \sqrt{516}}} + \frac{20+e}{25} \times \overline{\sqrt{405} + \frac{20e}{\sqrt{405}}}$$

$= 34$, that is, $\overline{400 + 20e} \times \overline{.044022 - }$ [illegible]$92e$ $+ \overline{20 + e} \times \overline{.804984 + .0398e} = 34$; whence rejecting e^2, &c. we have $1.713e = .1915$; and consequently $e = .1118$.

Thirdly, let there be given $\sqrt{1 - x} + \sqrt{1 - 2x^2} + \sqrt{1 - 3x^3} = 2$. Then, if $0.5 + e$ be substituted therein for x, it will become $\sqrt{0.5 - e} + \sqrt{0.5 - 2e} + \sqrt{0.625 - 2.25e} = 2$; or $\sqrt{0.5} - \sqrt{0.5} \times e + \sqrt{0.5} - \sqrt{0.5} \times 2e + \sqrt{.625} - \frac{2.25e}{2\sqrt{.625}} = 2$; whence 3.54[illegible] $= .204$, $e = .057$, and $x = 0,557$, with which the operation being repeated, the next value of x will come out $= .5516$.

Lastly, let there be given $\overline{1 + x}|^{\frac{1}{2}} + \overline{1 + x^2}|^{\frac{1}{3}} + \overline{1 + x^3}|^{\frac{1}{4}} = 6,5$. Here, by writing $3 + e$ for x, and proceeding as above, we shall have $2 + \frac{e}{4} + \overline{10}|^{\frac{1}{3}} + \frac{\overline{10}|^{\frac{1}{3}} \times 2e}{10} + \overline{28}|^{\frac{1}{4}} + \frac{\overline{28}|^{\frac{1}{4}} \times 27e}{4 \times 28} = 6.5$, that is, $6.455 + 1.23e = 6.5$; whence $e = .036$, and $x = 3.036$.

It may be observed that this method, as all the powers of e above the first are rejected, only doubles the number of places, at each operation: but, from what is therein shown, it is easy to see how it may be extended, so as to triple, or even quadruple, that number; but then the trouble, in every operation, would be increased in proportion, so that little or no advantage could be reaped therefrom.

Hitherto we have treated of equations which include one unknown quantity, only. If there be two equations given, and as many quantities (x and y) to be determined, one of those quantities must first be exterminated, [illegible] the two equations reduced to one, according to what is shown in sect. 9. But, if this cannot be readily done (which is sometimes the case) and the unknown quantities be so entangled as to render that way impracticable, the following method may be of use.

Let [illegible] values of x and y be assumed pretty near the truth (which, from the nature of the problem, may always be done); and let the values so assumed be denoted by f and g, and what they want of truth by s and t, respectively; that is, let $f + s = x$, and $g + t = y$; substitute these values in both equations, rejecting (by reason of their smallness) all the terms wherein more than one single dimension of the quantities s and t are concerned: let all the terms in the first equation, which are affected by s, be collected under their proper signs, and denoted by As; in like manner, let those affected by t be denoted by Bt; and those affected neither by s nor t, by Q: moreover, let the terms of the second equation, wherein [illegible] and t are concerned, be denoted by as and bt, respectively; and let the known terms, on the right-hand side of this equation, or those in which neither s nor t enters, be represented by q: Then the equations (be they of what kind they will) will stand thus, $As + Bt = Q$, and $as + bt = q$. By multiplying the former of which by b, and the latter by B, and then subtracting the one from the other, we shall have $bAs - Bas = bQ - Bq$; and therefore $s = \frac{bQ - Bq}{Ab - aB}$; whence $x \; (= f + s)$ is given.

Again, by multiplying the former equation by a, and the latter by A, &c. we shall have $aBt - Abt = aQ - Aq$, and therefore $t = \frac{aQ - Aq}{Ba - bA} = \frac{Aq - aQ}{Ab - aB}$; whence $y \; (= g + t)$ is likewise given.

It is easy to see that this method is also applicable, in cases of three[illegible] four equations, and as many unknow[illegible] quantities; but as these are cases that seldom occur in the resolution of problems, and, when they do, ar[illegible]ducible to those already considered, it will be needl[illegible]o take further notice of them here: I shall, therefore, content myself with giving an example or two, of the use of what is above laid down.

1. Let there be given $x^4 + y^4 = 10000$, a[illegible] $- y^5 = 25000$; to find x and y. Then, by wri[illegible] $+ s = x$, $g + t = y$, and proceeding according to the aforegoing directions, we shall have $f^4 + 4f^3s + g^4 + 4g^3t = 10000$, and $f^5 + 5f^4s - g^5 - 5g^4t = 25000$[illegible] or $4f^3s + 4g^3t = 10000 - f^4 -$ [illegible], and $5f^4s - 5g^4t = 25000 + g^5 - f^5$: the[illegible]re, in this case, $A = 4f^3$, $B = 4g^3$, $Q = 10000 - f^4 - g^4$, a[illegible] $5f^4$, $b = -5g^4$, and $q = 25000 + g^5 -$ [illegible] [illegible]ppears, from the first of the two give[illegible]uations, [illegible] x must be something less than 10, and [illegible]n the sec[illegible] that y mu[illegible] be less than x: I therefore take $f = 9$, and $g = 8$; and then A becomes $= 2916$, $B = 2048$, $Q = -657$, $a = 32805$, $b = -20480$, $q = -1281$; and therefore s $(\frac{bQ - Bq}{bA - Ba}) = -0.13$, and t $(\frac{Aq - aQ}{bA - Ba}) = -0.14$; hence $x = 8.87$, and $y = 7.86$, nearly.

Therefore, in order to repeat the operation, let f be now taken $= 8.87$, and $g = 7.86$; t[illegible]n will $A = 2791$, $B = 1942$, $Q = -6.76$, $a = 30950$, $b = -19083$, and $q = 94$; consequently s $(= \frac{bQ - Bq}{Ab - aB}) = .00047$, and t $(= \frac{Aq - aQ}{Ab - aB}) = -.00415$; whence $x = 8.87047$, and $y = 7.85585$; [illegible] which values are true to the last figure.

Example 2. Let there be given $\overline{20x + xy^2}\rceil^{\frac{1}{3}} + \overline{8x}\rceil^{\frac{1}{2}} = 12$, and $\sqrt{x^2 + y^2} + \frac{xy}{\sqrt{x^2 - y^2}} = 13$. Here the given equations, by writing $f + s$ for x, and $g + t$

for y, will become $\overline{20f + 20s + fg^2 + 2fgt + g^2s}\,\big|^{\frac{1}{3}} + \sqrt{8f + 8s} = 12$, and $\sqrt{f^2 + g^2 + 2fs + 2gt} + \dfrac{fg + ft + gs}{\sqrt{f^2 - g^2 + 2fs - 2gt}} = 13$: but $\overline{20f + fg^2 + 20s + 2fgt + g^2s}\,\big|^{\frac{1}{3}}$, by what is shown in p. 174, will be transformed to $\overline{20f + fg^2}\,\big|^{\frac{1}{3}} + \dfrac{\overline{20f + fg^2}\,\big|^{\frac{1}{3}}}{3 \times \overline{20f + fg^2}} \times \overline{20s + 2fgt + g^2s}$ (supposing all the terms that have more than one dimension of s and t, to be rejected, as inconsiderable); also $\sqrt{f^2 + g^2 + 2fs + 2gt}$, is transformed to $\sqrt{f^2 + g^2} + \dfrac{fs + gt}{\sqrt{f^2 + g^2}}$, and $\dfrac{1}{\sqrt{f^2 - g^2 + 2fs - 2gt}}$ to $\dfrac{1}{\sqrt{f^2 - g^2}} - \dfrac{fs - gt}{\overline{f^2 - g^2} \times \sqrt{f^2 - g^2}}$; therefore our equations will stand thus,

$$\overline{20f + fg^2}\,\big|^{\frac{1}{3}} + \frac{\overline{20f + fg^2}\,\big|^{\frac{1}{3}} \times \overline{20s + 2fgt + g^2s}}{3 \times \overline{20f + fg^2}} + \sqrt{8f} + \frac{4s}{\sqrt{8f}} = 12,$$

and

$$\sqrt{f^2 + g^2} + \frac{fs + gt}{\sqrt{f^2 + g^2}} + \overline{fg + ft + gs} \times \overline{\frac{1}{\sqrt{f^2 - g^2}} - \frac{fs - gt}{\overline{f^2 - g^2} \times \sqrt{f^2 - g^2}}} = 13:$$

which equations, if f be assumed $= 5$, and $g = 4$, will be reduced to $5.6462 + .01045 \times \overline{36s + 40t} + 6.3245 + .6324s = 12$, and $6.4031 + 781s + .625t + \overline{20 + 5t + 4s} \times \overline{.3333 - .1852s + .1482t} = 13$; whence $1.008s + .418t = .0293$, and $1.59s - 5.255t = .0698$: therefore, in this case, $A = 1.008$, $B = 0.418$, $Q = .0293$, $a = 1.59$, $b = -5.255$, and $q = .0698$: consequently s $\left(= \dfrac{bQ - Bq}{Ab - aB}\right) = 0.305$, and t $\left(= \dfrac{Aq - aQ}{Ab - aB}\right) = -.0040$; therefore $x = 5.0305$ and $y = 3.9960$.

SECTION XIII.

Of Indeterminate or Unlimited Problems.

A PROBLEM is said to be indeterminate, or unlimited, when the equations expressing the conditions thereof are fewer in number than the unknown quantities to be determined; such kinds of problems, strictly speaking, being capable of innumerable answers: but the answers in whole numbers, to which the question is commonly restrained, are, for the general part, limited to a determinate number; for the more ready discovering of which, I shall premise the following

LEMMA.

Supposing $\frac{ax \pm b}{c}$ to be an algebraic fraction, in its lowest terms, x being indeterminate, and a, b, and c given whole numbers; then, I say that the least integer, for the value of x, that will also give the value of $\frac{ax \pm b}{c}$ an integer, will be found by the following method of calculation.

Divide the denominator (c) by the coefficient (a) of the indeterminate quantity; also divide the divisor by the remainder, and the last divisor again by the last remainder; and so on, till a unit only remains.

Write down all the quotients in a line, as they follow; under the first of which write a unit, and under the second write the first; then multiply these two together, and having added the first term of the lower line (or a unit) to the product, place the sum under the third term of the upper line: multiply, in like manner, the next two corresponding terms of the two lines together, and, having added the second term of the lower to the product, put down the result under the fourth term of the upper one: proceed on, in this way, till you have multiplied by every number in the upper line.

Then, multiply the last number thus found by the absolute quantity (b) in the numerator of the given fraction, and divide the product by the denominator; so shall the remainder be the true value of x, required; provided the number of terms in the upper line be even, and the sign of b negative, or if that number be odd and the sign of b affirmative; but, if the number of terms be even, and the sign of b affirmative, or vice versa, *then the difference between the said remainder and the denominator of the fraction will be the true answer.*

In the general method here laid down, a is supposed less than c, and that these two numbers are *prime* to each other: for, were they to admit of a common measure, whereby b is not divisible, the thing would be impossible, that is, no integer could be assigned for x, so as to give the value of $\frac{ax \pm b}{c}$ an integer: the reason of which, as well as of the *lemma* itself, will be explained a little farther on: here it will be proper to put down an example or two, to illustrate the use of what has been already delivered.

Examp. 1. *Let the given quantity be* $\frac{87x - 50}{256}$.

Then the operation will stand as follows:

87)256(2
82)87(1   2, 1, 16, 2
5)82(16   1, 2, 3, 50, 103
2)5(2   50
1   256)5150(20
30 = x.

Examp. 2. *Given* $\frac{71x + 10}{89}$.

71)89(1
18)71(3   1, 3, 1
17)18(1   1, 1, 4, 5
1   10
50 = x.

Examp. 3. *Given* $\frac{377x - 250}{450}$.

377)450(1 1, 5, 6
73)377(5 1, 1, 6 37
12)73(6 250
1 1850
74
450)9250(20
250
450
200 = x.

Examp. 4. *Given* $\frac{987x + 651}{1235}$.

987)1235(1 1, 3, 1, 48, 1, 1
248)987(3 1, 1, 4, 5, 244, 249, 493
243)248(1 651
5)243(48 493
3)5(1 2465
2)3(1 2958
1 1235)320943(259
7394
12193
1078
1235
157 = x.

These four examples comprehend all the different cases that can happen with regard to the restrictions specified in the latter part of the rule. I shall now show the use thereof in the resolution of problems.

PROBLEM I.

To find the least whole number, which, divided by 17, *shall leave a remainder of* 7; *but, being divided by* 26, *the remainder shall be* 13.

Let x be the quotient, by 17, when 7 remains, or, which is the same thing, let $17x + 7$ express the number sought; then, since this number, when 13 is subtracted from it, is divisible by 26, it is manifest that $\frac{17x + 7 - 13}{26}$, or $\frac{17x - 6}{26}$ must be a whole number: whence, by proceeding according to the *lemma*, x will be found $= 8$; and consequently $17x + 7 = 143$, the number required. See the operation.

```
17)26(1
   17              1, 1, 1,
    9)17(1         1, 1, 2, 3
       9                    6
       8)9(1               18
         1                 26
                            8 = x.
```

PROBLEM II.

Supposing $9x + 13y = 2000$, *it is required to find all the possible values of* x *and* y *in whole positive numbers.*

By transposing $13y$, and dividing the whole equation by 9, we have $x = \frac{2000 - 13y}{9} = 222 - y + \frac{2 - 4y}{9}$; which, as x is a whole positive number, by the question, must also be a whole positive number, and so likewise $\frac{4y - 2}{9}$; from which the least value of y, in whole numbers, will come out $= 5$; and consequently the corresponding value of $x = 215$. From whence the rest of the answers, which are 16 in number, will be found, by adding 9 continually to the last value of y, and subtracting 13 from that of x, as in the annexed table, which exhibits all the possible answers in whole numbers.

$x =$	215	202	189	176	163	150	137	124	111	98	85	72	59	46	33	20	7
$y =$	5	14	23	32	41	50	59	68	77	86	95	104	113	122	131	140	149

In the same manner, the least value of y, and the greatest of x being found, in any other case, the rest of the answers will be obtained, by only adding the coefficient of x, in the given equation, to the last value of y, continually, and subtracting the coefficient of y from the corresponding value of x. Hence it follows, that, if the greatest value of x be divided by the coefficient of y, the remainder will be the least value of x, and that the quotient + 1 will give the number of all the answers. But it is to be observed, that, the equations here spoken of are such wherein the said coefficients are prime to each other; if this should not be the case, let the equation given be, first of all, reduced to one of this form, by dividing by the greatest common-measure.

PROBLEM III.

To find how many different ways it is possible to pay 100*l. in guineas and pistoles only; reckoning guineas at* 21 *shillings each, and pistoles at* 17.

Let x represent the number of guineas, and y that of the pistoles; then the number of shillings in the guineas being $21x$, and in the pistoles $17y$, we shall therefore have $21x + 17y = 2000$, and consequently $x = \frac{2000 - 17y}{21} = 95 + \frac{5 - 17y}{21}$; which being a whole number, by the question, it is manifest that $\frac{17y - 5}{21}$ must also be an integer: now the least value of y, in whole numbers, to answer this condition, will be found $= 4$, and the expression itself $= 3$; the corresponding, or greatest value of x being $= 92$; which being divided by 17, the coefficient of y (according to the preceding *note*), the quotient comes out 5, and the remainder 7; therefore the least value of x is 7, and the number of answers $(= 5 + 1)$ [illegible] these are as follows:

$x = 92$	75	58	41	24	7
$y = 4$	25	46	67	88	109.

PROBLEM IV.

To determine whether it be possible to pay 100*l. in guineas and moidores only; the former being reckoned at* 21 *shillings each, and the latter at* 2[illegible]

Here, by proceeding as in the last question, we have

$21x+27y=2000$; and consequently $x=\frac{2000-27y}{21}$ $=95-y-\frac{6y-5}{21}$: where, the fraction being in its least terms, and the numbers 6 and 21, at the same time, admitting of a common measure, a solution in whole numbers (*by the note to the preceding lemma*) is impossible. The reason of which depends on these two considerations: that, whatsoever number is divisible by a given number, must be divisible also by all the divisors thereof; and that any quantity which exactly measures the whole and one part of another, must do the like by the remaining part. Thus, in the present case, the quantity $6y-5$, to have the result a whole number, ought to be divisible by 21, and therefore divisible by 3, likewise, which is here a common measure of a and c: but $6y$, the former part of $6y-5$, is divisible by 3; therefore the latter part -5 ought also to be divisible by 3; which is not the case, and shows the thing proposed to be impossible.

PROBLEM V.

A butcher bought a certain number of sheep and oxen, for which he paid 100*l.*; *for the sheep he paid* 17 *shillings apiece, and for the oxen, one with another, he paid* 7 *pounds apiece; it is required to find how many he had of each sort.*

Let x be the number of sheep, and y that of the oxen; then, the conditions of the question being expressed in algebraic terms, we shall have this equation, *viz.* $17x+140y=2000$; and consequently $x=\frac{2000-140y}{17}=117-8y-\frac{4y-11}{17}$; which being a whole number, $\frac{4y-11}{17}$ must therefore be a whole number likewise: whence, by proceeding as above, we find $y=7$, and $x=60$; and this is the only answer the question will admit of; for the greatest value of x cannot in this case be divided by the coefficient of y, that is, 140 cannot be had in 60; and, therefore, according to the preceding note, the question can have only one answer, in whole numbers.

PROBLEM VI.

A certain number of men and women being merry-making together, the reckoning came to 33 *shillings, towards the discharging of which, each man paid* 3*s.* 6*d. and each woman* 1*s.* 4*d.: the question is, to find how many persons of both sexes the company consisted of.*

Let x represent the number of men, and y that of the women; so shall $42x + 16y = 396$, or $21x + 8y = 198$; and consequently $y = \frac{198 - 21x}{8} = 24 - 2x - \frac{5x - 6}{8}$: whence, y being a whole number, $\frac{5x - 6}{8}$ must likewise be a whole number; and the value of x, answering this condition, will be found $= 6$; and consequently that of y $(= 24 - 12 - 3) = 9$; which two will appear to be the only numbers that [illegible] answer the conditions of the question; because 21, [illegible]e coefficient of x, is here greater than 9, the greatest value of y.

PROBLEM VII.

One bought 12 *loaves for* 12 *pence, whereof some were two-penny ones, others penny ones, and the rest farthing ones: what number were there of each sort?*

Put $x =$ the number of the first sort, $y =$ that of the second, and $z =$ that of the third; and then, by the conditions of the question, we have these two equations, *viz.*

$$x + y + z = 12, \text{ and}$$
$$8x + 4y + z = 48.$$

Whereof the former being subtracted from the latter, in order to exterminate z, we thence get $7x + 3y = 36$, and therefore $y = \frac{36 - 7x}{3} = 12 - 2x - \frac{x}{3}$; whence it is evident that the value of $x = 3$, and consequently that $y = 5$, and $z = 4$; which are the numbers that were to be found.

PROBLEM VIII.

To find the least integer possible, which, being divided by 28, *shall leave a remainder of* 19; *but, being divided by* 19, *the remainder shall be* 15; *and, being divided by* 15, *the remainder shall be* 11.

First, to find the least whole number that can answer the two first conditions, let the quotient by 28, the first of the given divisors, be denoted by x, or, which is the same thing, let the said number be expressed by $28x + 19$; then this number, when 15 is subtracted from it, being divisible by 19, it is manifest that $\frac{28x+4}{19}$, or its equal $x + \frac{9x+4}{19}$, must be an integer; from whence the least value of x will be found $= 8$: and consequently $28x + 19 = 243$; which is the least whole number that can possibly satisfy the two first conditions. This being found, let the least number that is exactly divisible by both the said divisors 28 and 19, be now assumed; which, because 28 and 19 are prime to each other, will be equal to 28×19, or 532: then, since the number required, by the nature of the problem, must be some multiple of 532, increased by 243, it is plain that the said number may be represented by $532x + 243$; from which, if 11 be subtracted, and the remainder be divided by 15, the quotient $(\frac{532x+232}{15} = 35x + 15 + \frac{7x+7}{15})$ will be a whole number by the question, and consequently $\frac{7x+7}{15}$ a whole number also; from whence the least value of x will be found $= 14$, and consequently that of $532x + 243 = 7691$; which is the number that was to be found. In the same manner, the least number possible may be found, which, being successively divided by four or more given divisors, shall leave given remainders.

PROBLEM IX.

Supposing $87x + 256y = 15410$; *to determine the least value of* x, *and the greatest of* y, *in whole positive numbers.*

By transposition and division we have $y = \frac{15410 - 87x}{256} = 60 - \frac{87x - 50}{256}$: where the fraction being the same with that in *examp.* 1 to the premised *lemma*, the required value of x will be given from thence $= 30$; from thence that of y will likewise be known. But I shall in this place show the manner of deducing these values, independent of all previous considerations, by a method on which the demonstration of the *lemma* itself depends.

In order to this, it is evident, as the quantity $87x - b$ (supposing $b = 50$) is divisible by 256, that its double $174x - 2b$ must be likewise divisible by 256. But $256x$ is plainly divisible by 256; and if from this the quantity in the preceding line be subtracted, the remainder, $82x + 2b$, will be likewise divisible by the same number; since *whatsoever number measures the whole, and one part of another, must do the like by the remaining part:* for which reason, if the quantity last found be subtracted from the first, the remainder $5x - 3b$ will also be divisible by 256: and, if this new remainder, multiplied by 16, be subtracted from the preceding one (in order to farther diminish the coefficient of x), the difference $2x + 50b$ must be still divisible by the same number. In like manner, the double of the last line, or remainder, being subtracted from the preceding one, we have $x - 103b$, a quantity still divisible by 256: but $\frac{103b}{256} = 20 + \frac{30}{256}$; therefore $x - 30$ must be divisible by 256; and consequently x must be either equal to 30, or to 30 increased by some multiple of 256; but 30, being the least value, is that required.

It may not be amiss to add here another example, to illustrate the way of proceeding by this last method: wherein let us suppose the quantity given to be $\frac{987x + 651}{1235}$.

Then, making $b = 651$, the whole process will stand as follows:

From - -	$1235x$
sub. - - -	$987x + b$
1. rem. - -	$248x - b$
1. rem. × 3 -	$744x - 3b$
2. rem. - -	$243x + 4b$
3. rem. - -	$5x - 5b$
3. rem. × 48 -	$240x - 240b$
4. rem. - -	$3x + 244b$
5. rem. - - -	$2x - 249b$
6. rem. - -	$x + 493b$;

where, x being without a coefficient, let $493b$, or its equal 320943, be now divided by 1235, the common measure to all those quantities, and the remainder will be found 1078; therefore $x + 1078$ is likewise divisible by 1235; and consequently the least value of x $(= 1235 - 1078) = 157$. The manner of working, according to this method, may be a little varied; it being to the same effect, whether the last remainder, or a multiple of it, be subtracted from the preceding one, or the preceding one from some greater multiple of the last. Thus, in the example before us, the quantity $248x - b$, in the third line, might have been multiplied by 4, and the preceding one subtracted from the product; which would have given $5x - 5b$ (as in the sixth line) by one step less. If the manner of proceeding in these two examples be compared with the process for finding the same values, according to the *lemma*, the grounds of this will appear obvious.

PROBLEM X.

Supposing e, f, and g to denote given integers; to determine the value of x, such that the quantities $\frac{x-e}{28}$, $\frac{x-f}{19}$, and $\frac{x-g}{15}$, may all of them be integers.

By making $\frac{x-e}{28} = y$, we have $x = 28y + e$; which value being substituted in our second expression, it becomes $\frac{28y+e-f}{19}$; which, as well as y, is to be a whole number: but $\frac{28y+e-f}{19}$, by making $b = e - f$, will be $= y + \frac{9y+b}{19}$; and therefore $19y$ and $18y + 2b$ being both divisible by 19, their difference, $y - 2b$, must also be divisible by the same number; whence it is evident, that one value of y is $2b$; and that $2b + 19z$ (supposing z a whole number) will be a general value of y; and consequently that x $(= 28y + e) = 532z + 56b + e$ is a general value of x, answering the two first conditions. Let this, therefore, be substituted in the remaining expression $\frac{x-g}{15}$; which, by that means, becomes $\frac{532z+56b+e-g}{15} = 35z + 3b + \frac{7z+\beta}{15}$ (supposing $\beta = 11b + e - g = 12e - 11f - g$). Here $15z$ and $14z + 2\beta$ being both divisible by 15, their difference $z - 2\beta$ must likewise be divisible by the same number; and therefore one value of z will be 2β, and the general value of $z = 2\beta + 15w$: from whence the general value of x $(= 532z + 56b + e)$ is given $= 7980w + 1064\beta + 56b + e$; which, by restoring the values of b and β, becomes $7980w + 12825e - 11760f - 1064g$.

Now, to have all the terms affirmative, and their coefficients the least possible, let w be taken $= -e + 2f + g$; whence there results $4845e + 4200f + 6916g$, for a new value of x: from which, by expounding e, f, and g, by their given values, and dividing the whole by 7980, the least value of x, which is the remainder of the division, will be known.

PROBLEM XI.

If $5x + 7y + 11z = 224$; *it is required to find all the possible values of* x, y, *and* z, *in whole numbers.*

In this, and other questions of the same kind, where you have three or more indeterminate quantities, and only one equation, it will be proper, first of all, to find the limits of those quantities. Thus, in the present case, because x is $= \frac{224 - 7y - 11z}{5}$, and because the least values of y and z cannot, by [illegible] question, be less than unity, it is plain that x cannot be greater than $\frac{224 - 7 - 11}{5}$, or 41 : and, in the same manner, it will appear that y cannot be greater than 29, nor z greater than 19; which, therefore, are the required limits in this case. Moreover, since x is $= \frac{224 - 7y - 11z}{5} = 45 - y - 2z - \frac{1 + 2y + z}{5}$ = a whole number, it is manifest that $\frac{2y + z + 1}{5}$ must also be a whole number: let $z + 1$ be therefore considered as a known quantity, and let the same be represented by b, and then the last expression will become $\frac{2y + b}{5}$; from which, by proceeding as above, we shall get $y = 2b = 2z + 2$; whence the corresponding value of x comes out $= 42 - 5z$.

Let z be now taken $= 1$; then will $x = 37$ and $y = 4$; from the former of which values let the coefficient of y be continually subtracted, and to the latter, let that of x be continually added, and we shall thence have 37, 30, 23, 16, 9, and 2 for the successive values of x, and 4, 9, 14, 19, 24, and 29 for the corresponding values of y; which are all the possible answers when $z = 1$.

Let z be now taken $= 2$, then $x = 32$, and $y = 6$; let the former of these values be increased or decreased by the multiples of 7, and the latter by those of 5, as far as possible, till they become negative; so shall we have 39, 32, 25, 18, 11, and 4, for the successive values of x in this case, and 1, 6, 11, 16, 21, and 26, for the respective values of y: which are all the answers when $z = 2$.

Again, let z be taken $= 3$; then, by proceeding as above, the corresponding values of x and y will be found equal to 34, 27, 20, 13, 6; and 3, 8, 13, 18, 23, respectively; and so of the rest: whence we have the following answers, being 60 in numbers.

z	y	x
1	4. 9.14.19.24.29.	37.30.23.16. 9.2.
2	1. 6.11.16.21.26.	39.32.25.18.11.4.
3	3. 8.13.18.23.	34.27.20.13. 6.
4	5.10.15.20.25.	29.22.15. 8. 1.
5	2. 7.12.17.22.	31.24.17.10. 3.
6	4. 9.14.19.	26.19.12. 5.
7	1. 6.11.16.	28.21.14. 7.
8	3. 8.13.18.	23.16. 9. 2.
9	5.10.15.	18.11. 4.
10	2. 7.12.	20.13. 6.
11	4. 9.14.	15. 8. 1.
12	1. 6.11.	17.10. 3.
13	3. 8.	12. 5.
14	5.10.	14. 7.
15	2. 7.	9. 2.
16	4.	4.
17	1.	6.
18	3.	1.

PROBLEM XII.

If $17x + 19y + 21z = 400$; *it is proposed to find all the possible values of* x, y, *and* z, *in whole positive numbers.*

When the coefficients of the indeterminate quantities x, y, and z are nearly equal, as in this equation, it will be convenient to substitute for the sum of those quantities. Thus, let $x + y + z$ be put $= m$; then, by subtracting 17 times this last equation from the preceding one, we shall have $2y + 4z = 400 - 17m$; and by subtracting the given equation from 21 times the assumed one, $x + y + z = m$, there will remain $4x + 2y = 21m - 400$. Therefore, since y and z can have no values less than unity, it is plain, from the first of these two equations, that $400 - 17m$ cannot be less than 6, and therefore m not greater than $\frac{400 - 6}{17}$, or 23: also, because, by the second of the two last equations, $21m - 400$ cannot be less than 6, it is obvious that m cannot be less than $\frac{400 + 6}{21}$, or 19: therefore 19 and 23 are the limits of m in this case. These being determined, let $4x$ be transposed in the last equation, and the whole be divided by 2, and we shall have $y = 10m - 200 - 2x + \frac{m}{2}$: which being a whole number, by the question, $\frac{m}{2}$ must likewise be a whole number, and consequently m an even number; which, as the limits of m are 19 and 23, can only be 20, or 22: let, therefore, m be first taken $= 20$, then y will become $= 10 - 2x$ and z $(m - x - y) = 10 + x$; wherein x being taken equal to 1, 2, 3, and 4, successively, we shall have y equal to 8, 6, 4, 2, and z equal to 11, 12, 13, 14, respectively, which are four of the answers required. Again, let m be taken $= 22$; then will $y = 31 - 2x$, and $z = x - 9$; wherein let x be interpreted by 10, 11, 12, 13, 14, and 15, successively, whence y will come out 11, 9, 7, 5, 3, and 1;

and x equal to 1, 2, 3, 4, 5, and 6, respectively. Therefore we have the ten following answers; which are all the question admits of.

$x =$	1	2	3	4	10	11	12	13	14	15
$y =$	8	6	4	2	11	9	7	5	3	1
$z =$	11	12	13	14	1	2	3	4	5	6

PROBLEM XIII.

Supposing $7x + 9y + 23z = 9999$; *it is required to determine the number of all the answers, in positive integers.*

In cases like this, where the answers are very many, and the number of them only is required, the following method may be used.

In the general equation $ax + by + cz = k$ (where a and b are supposed prime to each other) let z be assumed $= 0$; and find the greatest value of x, and the least of y, in the equation $ax + by = k$, thence arising; denoting them by g and l: find, moreover, the least positive value of n, in whole numbers, from the equation $am + bn = c$, together with the corresponding value of m, whether positive or negative; then, supposing q to represent an integer, the general value of x may be expressed by $g - bq - mz$, and that of y by $l + aq - nz$; as will appear by substituting in the general expression $ax + by + cz$, which thereby becomes $ag - abq - amz + bl + abq - bnz + cz = k$, as it ought to be, because $ag + bl = k$, and all the rest of the terms destroy one another. And it may be observed farther, by the bye, and is evident from hence, that any two corresponding values of m and n, determined from the equation $am + bn = c$, will equally fulfil the conditions of the general equation; but the least are to be used, as being the most commodious. As to the limits of z and q, these are easily determined; the former from the original equation, and the latter from the general value of x; by which it appears that q cannot exceed $\frac{g - mz}{b}$; wherein the greatest, or the least value of z is to be used, according as the second term, after substitution for m, is ***positive*** or ***negative***. But, besides this, there is another limit, or particular value of q to be determined, which is of great use in finding the number of answers.

It is evident, from the given equations, that the values of x will begin to be negative, when z is so increased as to exceed $\frac{g - bq}{m}$; and that those of y will, in like manner, become negative, when z is taken greater than $\frac{l + aq}{n}$: therefore, as long as $\frac{g - bq}{m}$ continues greater than $\frac{l + aq}{n}$ (supposing the value of q to be varied) so long will x admit of a greater assumption for z than y will admit of, without producing negative values; and *vice versa*. By making, therefore, these two expressions equal to each other, the value of q will be given $\left(= \frac{ng - ml}{am + nb}\right) = \frac{ng - ml}{c}$; expressing the circumstance wherein both the values of x and y, by increasing z, become negative together. But this holds true only when m is a positive quantity; for, in the other case, the last term $(-mz)$ in the general value of x being positive, the particular values do not become negative by increasing, but by diminishing the value of z; it being evident, that no such can result from any assumption for z, but when q is greater than $\frac{g}{b}$.

To apply these observations to the equation, $7x + 9y + 23z = 9999$, proposed, we shall, in the first place, by taking $z = 0$, have $x = 1428 - y - \frac{2y - 3}{7}$: whence the least value of y is given $= 5$; and the greatest of $x = 1422$. Again, from the equation $am + bn = c$, or $7m + 9n = 23$, we have $m = 3 - n - \frac{2n - 2}{7}$; in which the least positive value of n is given $= 1$; and the corresponding value of $m = 2$; and so the general values of x and y do here become 1422 —

$9q - 2z$, and $5 + 7q - z$, respectively. From the former of which the greater limit of q is given $= \frac{1422 - 2}{9}$, or $157\frac{7}{9}$; and from $\frac{ng - ml}{c}$, expressing the lesser limit, we have 61, for the value of q, when the least value of x becomes equal to that of y. These limits being assigned, let q be now interpreted by 0, 1, 2, 3, 4, 5, &c. successively, up to 61, inclusive: whence the number of answers, or variations of y corresponding to every interpretation, will be found as in the margin. From whence it appears that the arithmetical progression $4 + 11 + 18 + 25 + 32$, &c. continued to 62 terms, will truly express the number of all the answers when q is less than 62: which number is therefore given $= \overline{4 + 61 \times 7 + 4} \times 31 = 13485$. In all which answers it is evident, that x, as well as y, will be positive (as it ought to be): because it has been proved that the least value of x, till q becomes $(= \frac{ng - ml}{c}) = 61\frac{2}{3}$, will be greater than that of y; which is positive, so far.

q	$y =$	N. Ans.
0	$5 - z$	4
1	$12 - z$	11
2	$19 - z$	18
3	$26 - z$	25
4	$33 - z$	32
&c.	&c.	&c.

But now, to find the answers when q is upwards of 61, we must have recourse to the general value of x; which, in these cases, by the different interpretations of z, becomes negative before that of y. Here, by beginning with the greatest limit, and writing 157, 156, 155, 154, &c. successively, in the room of q, it will appear, that the number of answers will be truly expressed by the series $4 + 8 + 13 + 17 + 22$, &c. continued to $157 - 61$ terms: which terms being united in pairs (because, in every two terms, the same fraction in the limit of z occurs) the

q	$x =$	$z \sqsupset$	N. Ans.
157	$9 - 2z$	$4\frac{1}{2}$	4
156	$18 - 2z$	9	8
155	$27 - 2z$	$13\frac{1}{2}$	13
154	$36 - 2z$	18	17
153	$45 - 2z$	$22\frac{1}{2}$	22
&c.	&c.	&c.	&c.

series $12 + 30 + 48 +$ &c. thence arising, will be a true arithmetical progression; whereof the common difference being 18, and the number of terms $= \frac{157 - 61}{2} = 48$, the sum will therefore be given $= 20880$: to which adding 13485, the number of answers when q was less than 62, the aggregate 34365 will be the whole number of all the answers required.

PROBLEM XIV.

To determine how many different ways it is possible to pay 1000l. without using any other coin than crowns, guineas, and moidores.

By the conditions of the problem we have $5x + 21y + 27z = 20000$; where taking $z = 0$, x is found $= 4000 - 4y - \frac{y}{5}$, and from thence the least value of $y = 0$ (0 being necessarily included here by the question): whence the greatest value of x is given $= 4000$. Moreover, from the equation $5m + 21n = 27$, we have $m = 5 - 4n - \frac{n-2}{5}$; from which $n = 2$, and $m = -3$: so that the general values of x and y, given in the preceding problem, will here become $4000 - 21q + 3z$, and $5q - 2z$. Moreover, from the given equation, the greatest limit of z appears to be $= \frac{20000}{27} = 740$; whence we also have $\frac{g - mz}{b} = \frac{4000 + 3 \times 740}{21} = 296 =$ the greatest limit of q; and $\frac{g}{b} = \frac{4000}{21} = 190$, expressing the lesser limit of q, when the value of x, answering to some interpretations of z, will become negative, while those of y still continue affirmative. To find the number of all these affirmative values, up to the greatest limit of q, let 0, 1, 2, 3, 4, 5, &c. be now written in the room of q (as in the margin). Whence it is evident that the said number is composed of the series 1 + 3 + 6 + 8 + 11 + 13, &c. continued to 297 terms; which terms (setting aside the first) being united in pairs, we shall have the arithmetical progression 9 + 19 + 29, &c. where the number of terms to be taken being 148, and common difference 10, the last term will therefore be 1479, and the sum of the whole progression 110112: to which adding (1) the term omitted, we have 110113, for the number of all the answers, including those wherein the value of x is negative; which last must therefore be found and deducted.

q	$y =$	Quot.	N. Ans.
0	$0 - 2z$	0	1
1	$5 - 2z$	$2\frac{1}{2}$	3
2	$10 - 2z$	5	6
3	$15 - 2z$	$7\frac{1}{2}$	8
4	$20 - 2z$	10	11
5	$25 - 2z$	$12\frac{1}{2}$	13
&c.	&c.	&c.	&c.

In order to this we have already found, that these negative values do not begin to have place till q is greater than 190: let, therefore, 191, 192, 193, &c. be substituted, successively, for q; from whence it will appear that the number of all the said negative values is truly exhibited by the arithmetical progression 4 + 11 + 18 + 25, &c. continued to 296 — 190 terms; whereof the sum is 39379; which subtracted from 110113, found above, leaves 70734, for the number of answers required.

q	$x.$	Quot.	N. Ans.
191	$3z - 11$	$3\frac{2}{3}$	4
192	$3z - 32$	$10\frac{2}{3}$	11
193	$3z - 53$	$17\frac{2}{3}$	18
194	$3z - 74$	$24\frac{2}{3}$	25
&c.	&c.	&c.	&c.

After the manner of these two examples (which illustrate the two different cases of the general solution, given in the preceding problem) the number of answers may be found in other equations, wherein there are three indeterminate quantities. But, in summing up the numbers arising from the different interpretations of q, due regard must be had to the fractions exhibited in the third column expressing the limits of z; because, to have a regular progression, the terms of the series in the fourth column, exhibiting the number of answers, must be united by twos, threes, or fours, &c. according as one and the same fraction occurs every second, third, or fourth, &c. term (the odd terms, when there happen any thing over, being always to be set aside, at the beginning of the series). And it may be observed farther, that, to determine the sum of the progression thus arising, it will be sufficient to find the first term only, by an actual addition: since not only the number of terms, but the common difference also, will be known; being always equal to the common difference of the limits of z (or of the quotients in the said third column) multiplied by the square of the number of terms united into one; whereof the reason is evident. But all this relates to the cases wherein the coefficients of the indeterminate quantities, in the given equations, are (two of them at least) prime to each other: I shall add one example more, to show the way of proceeding when those coefficients admit of a common measure.

PROBLEM XV.

Supposing $12x + 15y + 20z = 100001$; *it is required to find the number of all the answers in positive integers.*

It is evident, by transposing $20z$ and dividing by (3) the greatest common measure of x and y, that $4x + 5y$, and consequently its equal $33333 - 6z - \frac{2z - 2}{3}$, must be an integer, and therefore $2z - 2$ divisible by 3: but $3z$ is divisible by 3, and so the difference of these two, which is $z + 2$, must be likewise divisible by the same number, and consequently $z = 1 +$ some multiple of 3. Make, therefore, $1 + 3u = z$ (u being an integer); then the given equation, by substituting this value, will become $12x + 15y + 60u + 20 = 100001$; which, by division, &c. is reduced to $4x + 5y + 20u = 33327$: wherein the coefficients of x and y are now prime to each other, and we are to find the number of all the variations, answering to the different interpretations of u, from 0 to the greatest limit inclusive.

By proceeding, therefore, as in the aforegoing cases, we have $x = 8331 - y - \frac{y-3}{4}$; whence the least value of y is given $= 3$, and the greatest of $x = 8328$. Moreover, from the equation $4m + 5n = 20$, we have $m = 5 - n - \frac{n}{4}$; whence $n = 0$, and $m = 5$. Therefore the general values of x and y (*given in problem* 13) do here become $8328 - 5q - 5u$, and $3 + 4q$; from the former of which the greatest limit of q is given $= \frac{8328}{5} = 1665$. Now, since the value of y will here continue positive, in all substitutions for q and u, as no negative quantity enters therein, the whole numbers of answers will be determined by the values of x alone.

In order to this, let q be successively expounded by 1665, 1664, 1663, &c. and it will thence appear that the said number will be truly defined by 1666 terms of the arithmetical progression $1 + 2 + 3 + 4 + 5$, &c. whereof the sum is found to be 1388611.

q	x	Quot.	N. Ans.
1665	$3 - 5u$	$0\frac{3}{5}$	1
1664	$8 - 5u$	$1\frac{3}{5}$	2
1663	$13 - 5u$	$2\frac{3}{5}$	3
&c.	&c.	&c.	&c.

When there are four indeterminate quantities in the given equation, the number of all the answers may be determined by the same methods; for any one of those quantities may be interpreted by all the integers, successively, up to its greatest limit, which is easily determined; and the number of answers corresponding to each of these interpretations may be found as above; the aggregate of all which will consequently be the whole number of answers required: which sum or aggregate may, in many cases, be derived by the methods given in section 14, for summing of series by means of a known relation of their terms.

But this being a matter of more speculation than real use, I shall now pass on to other subjects.

SECTION XIV.

The Investigation of the Sums of the Powers of Numbers in Arithmetical Progression.

BESIDES the two sorts of progressions treated of in section 10, there are infinite varieties of other kinds; but the most useful, and the best known, are those consisting of the powers of numbers in arithmetical progression; such as $1^2 + 2^2 + 3^2 + 4^2 \ldots\ldots n^2$, and $1^3 + 2^3 + 3^3 + 4^3 \ldots\ldots n^3$, *&c.* where n denotes the number of terms to which each progression is to be continued. In order to investigate the sum of any such progression, which is the design of this section, it will be requisite, first of all, to premise the following

LEMMA.

If any expression, or series, as $\left.\begin{array}{l} An + Bn^2 + Cn^3 + Dn^4 \text{ \&c.} \\ - an - bn^2 - cn^3 - dn^4 \text{ \&c.} \end{array}\right\}$, involving the powers of an indeterminate quantity n, be universally equal to nothing, whatsoever be the value of n; then, I say, the sum of the coefficients $A - a$, $B - b$, $C - c$, &c. of each rank of homologous terms, or of the same powers of n, will also be equal to nothing.

For, in the first place, let the whole equation $\left.\begin{array}{l} An + Bn^2 + Cn^3 \text{ \&c.} \\ - an - bn^2 - cn^3 \text{ \&c.} \end{array}\right\} = 0$, be divided by n, and we shall have $\left\{\begin{array}{l} A + Bn + Cn^2 \text{ \&c.} \\ - a - bn - cn^2 \text{ \&c.} \end{array}\right\} = 0$; and this being universally so, be the value of n what it will: let, therefore, n be taken $= 0$, and it will become $\left\{\begin{array}{c} A \\ - a \end{array}\right\} = 0$; which being rejected as such, out of the last equation, we shall next have $\left.\begin{array}{l} + Bn + Cn^2 + Dn^3 \text{ \&c.} \\ - bn - cn^2 - dn^3 \text{ \&c.} \end{array}\right\} = 0$; whence, dividing again by n, and proceeding in the very same manner, $B - b$ is also proved to be $= 0$; and from thence $C - c$, $D - d$, &c. Q. E. D.

Now, to apply what is here demonstrated to the purpose above specified, it will be proper to observe, first, that as the value of any progression, $1^2 + 2^2 + 3^2 + 4^2 \ldots\ldots n^2$, varies according as n, the number of its terms, varies, it must (if it can be expressed in a general manner) be explicable by n, and its powers with determinate coefficients; secondly, it is obvious that those powers, in the cases above proposed, must be rational, or such whose indices are whole positive numbers; because the progression, being an aggregate of whole numbers, cannot admit of surd quantities; lastly, it will appear that the greatest of the said indices cannot exceed the common index of the progression by more than unity; for, otherwise, when n is taken indefinitely great, the highest power of n would be indefinitely greater than all the rest of the terms put together.

Thus, the highest power of n, in an expression universally exhibiting the value of $1^2 + 2^2 + 3^2 \ldots\ldots n^2$, cannot be greater than n^3; for $1^2 + 2^2 + 3^2 \ldots\ldots n^2$ is manifestly less than n^3 (or $n^2 + n^2 + n^2 +$ &c. continued to n terms); but n^4, when n is indefinitely great, is indefinitely greater than n^3, or any other inferior power of n, and therefore cannot enter into the equation. This being premised, the method of investigation may be as follows.

Case 1°. *To find the sum of the progression* $1 + 2 + 3 + 4 \ldots. n$.

Let $An^2 + Bn$ be assumed, according to the foregoing observations, as a universal expression for the value of $1 + 2 + 3 + 4 \ldots\ldots n$; where A and B represent unknown, but determinate quantities. Therefore, since the equation is supposed to hold universally true, whatsoever be the number of terms, it is evident, that, if the number of terms be increased by unity, or, which is the same thing, if $n + 1$ be written therein, instead of n, the equality will still subsist, and we shall have $A \times \overline{n+1}|^2 + B \times \overline{n+1} = 1 + 2 + 3 + 4 \ldots\ldots\ldots n \times \overline{n+1}$. From which the first equation being subtracted, there remains $A \times \overline{n+1}|^2 - An^2 + B \times \overline{n+1} - Bn = n + 1$: this contracted will be $2An + A + B = n + 1$: whence we have $\overline{2A - 1} \times n + A + B - 1 = 0$: wherefore, by taking $2A - 1 = 0$, and $A + B - 1 = 0$ (*according to the lemma*), we have $A = \frac{1}{2}$, and $B = \frac{1}{2}$; and consequently $1 + 2 + 3 + 4 \ldots\ldots n$ $(= An^2 + Bn) =$ $\frac{n^2}{2} + \frac{n}{2}$, or $\frac{n \times \overline{n+1}}{2}$.*

Case 2°. *To find the sum of the progression* $1^2 + 2^2 + 3^2 \ldots\ldots n^2$, or $1 + 4 + 9 + 16 \ldots\ldots n^2$.

Let $An^3 + Bn^2 + Cn$, according to the aforesaid observations, be assumed $= 1^2 + 2^2 + 3^2 + 4^2 \ldots\ldots n^2$: then, by reasoning as in the preceding case, we shall have $A \times \overline{n+1}|^3 + B \times \overline{n+1}|^2 + C \times \overline{n+1} = 1^2 + 2^2 + 3^2 + 4^2 \ldots\ldots n^2 + \overline{n+1}|^2$; that is, by involving $n + 1$ to its several powers, $An^3 + 3An^2 + 3An + A + Bn^2 + 2Bn + B + Cn + C = 1^2 + 2^2 + 3^2 + 4^2 \ldots\ldots n^2 + \overline{n+1}|^2$: from which, subtracting the former equation, we get $3An^2 + 3An + A + 2Bn + B + C$ $(= \overline{n+1}|^2) = n^2 + 2n + 1$; and consequently $\overline{3A - 1} \times n^2 + \overline{3A + 2B - 2} \times n + A + B + C - 1 = 0$; whence (*by the lemma*), $3A - 1 = 0$, $3A + 2B - 2 = 0$, and $A + B + C - 1 = 0$; therefore $A = \frac{1}{3}$, $B = \frac{2 - 3A}{3} = \frac{1}{2}$, $C = 1 - A - B = \frac{1}{6}$; and consequently $1 + 4 + 9 + 16 \ldots\ldots n^2 = \frac{n^2}{3} + \frac{n^3}{2} + \frac{n}{6}$, or $\frac{n \cdot \overline{n+1} \cdot \overline{2n+1}}{6}$.

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* In this investigation, it is taken for granted, that the sum of the progression is capable of being exhibited by means of the powers of n, with proper coefficients: which assumption is verified by the process itself: for it is evident from thence, that the quantities $An^2 + Bn$, and $1 + 2 + 3 + 4 \ldots\ldots n$, under the values of A and B there determined, are always increased equally, by taking the value of n greater by a unit: if, therefore, they be equal to each other, when n is $= 0$ (as they actually are), they must also be equal when n is 1; and so, likewise, when n is 2, &c. &c. And the same reasoning holds good in all the following cases.

Case 3°. *To determine the sum of the progression* $1^3 + 2^3 + 3^3 + 4^3 \ldots\ldots n^3$, or $1 + 8 + 27 + 64 \ldots\ldots n^3$.

By putting $An^4 + Bn^3 + Cn^2 + Dn = 1 + 8 + 27 + 64 \ldots\ldots n^3$, and proceeding as above, we shall have $4An^3 + 6An^2 + 4An + A + 3Bn^2 + 3Bn + B + 2Cn + C + D$ $(= \overline{n+1}|^3) = n^3 + 3n^2 + 3n + 1$; and therefore $\overline{4A - 1} \times n^3 + \overline{6A + 3B - 3} \times n^2 + \overline{4A + 3B + 2C - 3} \times n + A + B + C + D - 1 = 0$: hence $A = \frac{1}{4}$, $B \left(= \frac{3 - 6A}{3}\right) = \frac{1}{2}$, $C \left(= \frac{3 - 4A - 3B}{2}\right) = \frac{1}{4}$, $D (= 1 - A - B - C) = 0$; and therefore $1^3 + 2^3 + 3^3 + 4^3 \ldots\ldots n^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$, or $\frac{n^2 \times \overline{n+1}|^2}{4}$. In the very same manner it will be found that

$$1^4 + 2^4 + 3^4 \ldots\ldots n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$

$$1^5 + 2^5 + 3^5 \ldots\ldots n^5 = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}.$$

$$1^6 + 2^6 + 3^6 \ldots\ldots n^6 = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}.$$

&c. &c.

In order to exemplify what has been thus far delivered, let it, in the first place, be required to find the sum of the series of squares $1 + 4 + 9 + 16$, *&c.* continued to 10 terms: then, by substituting 10 for n, in the general expression $\frac{n \cdot \overline{n+1} \cdot \overline{2n+1}}{6}$ (or $\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$), found by case 2°, there will come out 385, for the required sum of the progression: which, the number of terms being here small, may be easily confirmed, by actually adding the 10 terms together. Secondly, let it be required to find the number of cannon-shot in a square pile, whose side is 50. Here, by writing 50 for n in the same expression, $\frac{n \cdot \overline{n+1} \cdot \overline{2n+1}}{6}$, we shall have $\left(\frac{50 \times 51 \times 101}{6}\right)$ 42925, expressing the number of shot in such a pile. Lastly, suppose a pyramid composed of 100 stones of a cubical figure; whereof the length of the side of the highest is one inch; of the second two inches; of the third three inches, *&c.* Here, by writing 100 instead of n, in the third general expression, we have 25502500, for the number of solid inches in such a pyramid.

Hitherto, regard has been had to such progressions as have unity for their first term, and likewise for the common difference; but the same equations, or theorems, with very little trouble, may be also extended to those cases where the first term, and the common difference, are any given numbers, provided the former of them be any multiple of the latter. Thus, suppose it were required to find the sum of the progression $6^2 + 8^2 + 10^2$ *&c.* (or $36 + 64 + 100$ *&c.*) continued to eight terms: then, by making (4) the square of the common difference a general multiplicator, the given expression will be reduced to $4 \times \overline{3^2 + 4^2 + 5^2 \ldots\ldots 10^2}$: but the sum of the progression $1^2 + 2^2 + 3^2 + 4^2 \ldots\ldots 10^2$ is found, by the second theorem, to be 385; from which, if (5) the sum of the two first terms (which the series $3^2 + 4^2 + 5^2 \ldots\ldots 10^2$ wants) be taken away, the remainder will be 380; and this, multiplied by 4, gives 1520, for the true sum of the proposed progression: and so of others.

But if the first term be not divisible by the common difference, as in the progression, $5^2 + 7^2 + 9^2$ &c. the speculation is a little more difficult; nevertheless, the sum of the series, in any such case, may still be found, from the same theorems.

Let the series $\overline{m+e}|^2 + \overline{m+2e}|^2 + \overline{m+3e}|^2 \ldots\ldots \overline{m+ne}|^2$ be proposed, where m and e denote any quantities whatever, and where n represents the number of terms. Then, by actually raising each root to its second power, and placing the terms in order, the given expression will stand thus:

$$\left.\begin{array}{l} m^2 + m^2 + m^2 \ldots\ldots m^2 \\ 2me + 4me + 6me \ldots\ldots 2nme \\ e^2 + 4e^2 + 9e^2 \ldots\ldots n^2e^2 \end{array}\right\}.$$

Now, it is evident that the sum of the first rank, or series, is $n \times m^2$: also the sum of the second, or $2me \times \overline{1+2+3+4\ldots\ldots n}$, appears (*by case* 1) to be $2me \times \frac{n \times \overline{n+1}}{2}$; and that of the third, or $e^2 \times \overline{1+4+9+16\ldots\ldots n^2}$ (*by case* 2) $= e^2 \times \frac{n \cdot \overline{n+1} \cdot \overline{2n+1}}{6}$: therefore the sum of the whole progression, $\overline{m+e}|^2 + \overline{m+2e}|^2 + \overline{m+3e}|^2 \ldots\ldots \overline{m+ne}|^2$ is $= n \cdot m^2 + n \cdot \overline{n+1} \cdot me + \frac{n \cdot \overline{n+1} \cdot \overline{2n+1} \cdot e^2}{6}$.

In like manner, if the series proposed be $\overline{m+e}|^3 + \overline{m+2e}|^3 + \overline{m+3e}|^3 \ldots\ldots \overline{m+ne}|^3$; then may it be resolved into

$$\left\{\begin{array}{l} \overline{1+1+1\ldots\ldots1} \times \overline{m^3} \\ \overline{1+2+3\ldots\ldots n} \times \overline{3m^2e} \\ \overline{1+4+9\ldots\ldots n^2} \times \overline{3me^2} \\ \overline{1+8+27\ldots\ldots n^3} \times \overline{e^3} \end{array}\right\}:$$

whose sum, by

the forementioned theorems, will appear to be $n \cdot m^3 + \frac{n \cdot \overline{n+1} \cdot 3m^2e}{2} + \frac{n \cdot \overline{n+1} \cdot \overline{2n+1} \cdot me^2}{2} + \frac{n^2 \cdot \overline{n+1}|^2 \cdot e^3}{4}$. And, by following the same method, the sums of other series may be determined, not only of powers, but likewise of rectangles, and solids, &c. provided their sides, or factors, be in arithmetical progression. Thus, for example, let there be proposed the series of rectangles $\overline{m+e} \cdot \overline{p+e} + \overline{m+2e} \cdot \overline{p+2e} + \overline{m+3e} \cdot \overline{p+3e} \ldots\ldots + \overline{m+ne} \cdot \overline{p+ne}$. Then, the factors being actually multiplied together, and the terms placed in order, the given series will be resolved into the three following ones:

$$mp + mp + mp + mp \ldots\ldots + mp$$
$$\overline{m+p} \cdot e + \overline{m+p} \cdot 2e + \overline{m+p} \cdot 3e + \overline{m+p} \cdot 4e \ldots\ldots + \overline{m+p} \cdot ne$$
$$e^2 + 4e^2 + 9e^2 + 16e^2 \ldots\ldots + n^2e^2.$$

Whereof the respective sums (*by case* 1 and 2) are $mp \times n$, $\overline{m+p} \cdot e \times \frac{n \cdot \overline{n+1}}{2}$, and $e^2 \times \frac{n \cdot \overline{n+1} \cdot \overline{2n+1}}{6}$ and the aggregate of all these, or $n \times \overline{mp + \frac{n+1}{2} \cdot \overline{m+p} \cdot e + \frac{\overline{n+1} \cdot \overline{2n+1}}{6} \cdot e^2}$, is consequently the true sum of the series of rectangles proposed.

From this last general expression, the number of cannon-shot in an oblong pile, whether whole or broken, will be known. For, supposing $e = 1$, our series of rectangles becomes $\overline{m+1} \cdot \overline{p+1} + \overline{m+2} \cdot \overline{p+2} + \overline{m+3} \cdot \overline{p+3} \ldots\ldots + \overline{m+n} \cdot \overline{p+n}$; and the sum thereof $= n \times \overline{mp + \frac{n+1}{2} \cdot \overline{m+p} + \frac{\overline{n+1} \cdot \overline{2n+1}}{6}} =$ the number sought: where $m+1$ and $p+1$ represent the length and breadth of the uppermost rank, or tire: n being the number of ranks one above another.

But the expression here brought out may be reduced to $\frac{n}{4} \times \overline{\overline{2m+n+1} \;.\; \overline{2p+n+1} + \frac{\overline{n+1} \;.\; \overline{n-1}}{3}}$; which is better adapted to practice, and which, expressed in words, gives the following rule.

To twice the length, and to twice the breadth of the uppermost rank, add the number of ranks less one, and multiply the two sums together; also multiply the number of ranks less one, by that number more one, and add $\frac{1}{3}$ of this product to the former; then $\frac{1}{4}$ of the sum multiplied by the number of ranks will be the answer.

As a rule of this sort is of frequent use to persons concerned in artillery, it may not be improper to add an example or two, by way of illustration.

1. Suppose a complete pile, consisting of 15 tires, or ranks, and suppose the number of shot in the uppermost (which, in this case, is a single row) to be 32. Then the first product mentioned in the rule will be $\overline{64+14} \times \overline{2+14} = 78 \times 16 = 1248$; and the second $= 14 \times 16 = 224$; $\frac{1}{3}$ whereof is $74\frac{2}{3}$, and this, added to 1248, gives $1322\frac{2}{3}$; whereof $\frac{1}{4}$ part is $330\frac{2}{3}$; which, multiplied by 15, gives 4960, for the whole number of shot in such a pile.

2. Let the pile be a broken one, such, that the length and breadth of the uppermost tire may be 25 and 16, and the number of tires 11.

Here, we have $\overline{50+10} \times \overline{32+10} = 60 \times 42 = 2520$, for the first product; and $12 \times 10 = 120$, for the second: therefore $\frac{2560}{4} \times 11 = 640 \times 11 = 7040$, is the true answer.

Having exemplified the use of the theorem, for finding the sum of a series of rectangles, I shall here subjoin one instance of that preceding it, for determining the sum of a series of cubes; wherein the value of the first 10 terms of the progression $\overline{2+\sqrt{2}}|^3 + \overline{3+2\sqrt{2}}|^3 + \overline{4+3\sqrt{2}}|^3 + \overline{5+4\sqrt{2}}|^3$ &c. is required. Here,

e being $1 + \sqrt{2}$, m will be $= 1$; therefore, by writing 10, 1, and $1 + \sqrt{2}$ for n, m, and e, respectively, in the general expression, it will become $10 + \frac{10.11.3.\overline{1+\sqrt{2}}}{2}$ $+ \frac{10 . 11 . 21 . \overline{1 + \sqrt{2}}]^2}{2} + \frac{100 . 121 . \overline{1 + \sqrt{2}}]^3}{4} =$ $24815 + 17600\sqrt{2}$, the value sought.

If any one be desirous to see this speculation carried further, so as to extend to series of powers, whose indices are fractions; such as square roots, cube roots, *&c.* I must beg leave to refer to my *Essays*, where it is treated in a general manner. Here, I must desire the reader to observe, once for all, that the theorems above found will hold equally true, in case of a descending series, such as $\overline{m - e}]^2 + \overline{m - 2e}]^2$ *&c.* or $\overline{m - e}]^3 + \overline{m - 2e}]^3$ *&c.* provided the signs of the second, fourth, *&c.* terms be changed; as is evident from the investigation.

Although the subject of this section has, already, been pretty largely insisted on, yet it may not be improper to add a different method, whereby the same conclusions will, in many cases, be more easily derived: in order to which it is necessary to premise the subsequent

LEMMA.

If $a + b + c + d + e +$ *&c.* be a series, whereof the terms, a, b, c, d, &c. are so related to each other, that the sum, or value thereof, can be universally expounded by an expression of this form, *viz.* $An + B \times n \times \overline{n - 1} + C \times n \times \overline{n - 1} \times \overline{n - 2} + D \times n \times \overline{n - 1} \times \overline{n - 2} \times \overline{n - 3}$ *&c.* n being the number of terms to which the series is to be continued, and A, B, C, D, *&c.* determinate coefficients; then, I say, the values of those coefficients will be as hereunder specified, *viz.*

$$A = a$$

$$B = \frac{-a + b}{2},$$

$$C = \frac{a - 2b + c}{2 . 3},$$
$$D = \frac{-a + 3b - 3c + d}{2 . 3 . 4},$$
$$E = \frac{a - 4b + 6c - 4d + e}{2 . 3 . 4 . 5},$$
&c. *&c.*

For, since the equation $A \times n + B \times n \times \overline{n-1} + C \times n \times \overline{n-1} \times \overline{n-2} + D \times n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}$ *&c.* $= a + b + c + d + e$, &c. is supposed to hold universally true, let the number of terms be what it will, let n be expounded by 1, 2, 3, 4, *&c.* successively, and the general equation will become

1°. $*A = a$,
2°. $2A + 2B = a + b$,
3°. $3A + 6B + 6C = a + b + c$,
4°. $4A + 12B + 24C + 24D = a + b + c + d$,
5°. $5A + 20B + 60C + 120D + 120E = a + b + c + d + e$,
&c. &c.

Now, the double of the first of these equations being subtracted from the second, its triple from the third, and its quadruple from the fourth, *&c.* we shall have

$*2B = b - a$,
$6B + 6C = -2a + b + c$,
$12B + 24C + 24D = -3a + b + c + d$,
$20B + 60C + 120D + 120E = -4a + b + c + d + e$,
&c. &c.

Again, if the triple of the first of these be subtracted from the second, and its sextuple from the third, *&c.* we shall next have

$*6C = a - 2b + c$,
$24C + 24D = 3a - 5b + c + d$,
$60C + 120D + 120E = 6a - 9b + c + d + e$.

Moreover, by taking the quadruple of the first of these from the second, *&c.* we get

$*24D + -a + 3b - 3c + d$, and
$120D + 120E = -4a + 11b - 9c + d + e$;

from the latter of which subtract the quintuple of the former, and there will remain

$$*120E = a - 4b + 6c - 4d + e.$$

Now, divide each of the equations marked thus (*), by the coefficient of its first term, and there will come out the very values of A, B, C, D, *&c.* above exhibited. Q. E. D.

COROLLARY.

If every term of the proposed series, a, b, c, d, &c. be subtracted from the next following, the first of the remainders, $-a+b$, $-b+c$, $-c+d$, $-d+e$, &c. divided by 2, gives the value of B, the coefficient of the second term of the assumed series. And if each of the quantities thus arising be subtracted from its succeeding one, the first of the new remainders, $a-2b+c$, $b-2c+d$, $c-2d+e$, &c. divided by 6, will be equal to C, the coefficient of the third term of the same series. In like manner, if each of these last remainders be again subtracted from its succeeding one, the next remainders will be $-a+3b-3c+d$, $-b+3c-3d+e$, &c. whereof the first, divided by 24, gives the coefficient of the fourth term, *&c. &c.* Therefore, if the first remainder of the first order be denoted by P, the first of the second order by Q, the first of the third by R, the first of the fourth by S, *&c.* then $\frac{P}{2}$ being $= B$, $\frac{Q}{2\,.\,3} = C$, $\frac{R}{2\,.\,3\,.\,4} = D$, $\frac{S}{2\,.\,3\,.\,4\,.\,5} = E$, *&c.* it is manifest that the sum of the series $a+b+c+d+e+f$, &c. will be truly expressed by

$$an + P \times \frac{n \times \overline{n-1}}{1\,.\,2} + Q \times \frac{n \times \overline{n-1} \times \overline{n-2}}{1\,.\,2\,.\,3} +$$

$$R \times \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{1\,.\,2\,.\,3\,.\,4} +$$

$$S \times \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{1\,.\,2\,.\,3\,.\,4\,.\,5}, \text{\&c.}$$

Example 1. Let the sum of the series of squares $1+4+9+16 \,.\,.\,.\,.\, + n^2$ be required. Then, taking the differences of the several orders, according to the preceding corollary, we have

1, 4, 9, 16, 25, 36, &c.
3, 5, 7, 9, 11, &c.
2, 2, 2, 2, &c.
0, 0, 0, &c.

Therefore a in this case being $= 1$, $P = 3$, $Q = 2$, and R, S, *&c.* each $= 0$, the sum of the whole series, $1 + 4 + 9 + 16 + 25 \ldots n^2$, is found $= n + \frac{3n \times \overline{n-1}}{2} + \frac{n \times \overline{n-1} \times \overline{n-2}}{3} = \frac{2n^3 + 3n^2 + n}{6} = \frac{n \times \overline{n+1} \times \overline{2n+1}}{6}$.

Example 2. Let it be required to find the sum of n terms of the following series of cubes, *viz.* $27 + 64 + 125 + 216 + 343 + 512$, *&c.* Proceeding here as in the last example, we have

27, 64, 125, 216, 343, 512, &c.
37, 61, 91, 127, 169, &c.
24, 30, 36, 42, &c.
6, 6, 6, &c.
0, 0, &c.

Therefore, by substituting 27 for a, 37 for P, 24 for C, and 6 for D, we thence get
$27n + \frac{37n \times \overline{n-1}}{2} + \frac{24n \times \overline{n-1} \times \overline{n-2}}{2 . 3} + \frac{6n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{1 . 2 . 3 . 4}$; which, abbreviated, becomes $\frac{n^4}{4} + \frac{5n^3}{2} + \frac{37n^2}{4} + 15n$, the sum, or value, required.

Example 3. Let the series propounded be $2 + 6 + 12 + 20 + 30$, *&c.* In this case, we have

2, 6, 12, 20, 30, &c.
4, 6, 8, 10, &c.
2, 2, 2, &c.

Hence, a being $= 2$, $P = 4$, $Q = 2$, and R, S, *&c.* each $= 0$, the sum of the series will therefore be

$$2n + \frac{4n \times \overline{n-1}}{2} + \frac{2n \times \overline{n-1} \times \overline{n-2}}{6} = \frac{n^3 + 3n^2 + 2n}{3}$$

$$= \frac{n \times \overline{n+1} \times \overline{n+2}}{3}.$$

And, in the same manner, the sum of the series may be truly found, in all cases where the differences of any order become equal among themselves: and even in other cases, where the differences do not terminate, a near approximation may be obtained, by carrying on the process to a sufficient length.

SECTION XV.

Of Figurate Numbers, their sums, and the sums of their reciprocals, with other matters of the like nature.

THAT series which arises by adding together a rank

of	Units (called fig. No. of the 1st ord.)	is called a series of fig. numb. of the	2d	order.
	Figurate numbers of the 2d order		3d	
	Figurate numbers of the 3d order		4th	
	Figurate numbers of the 4th order		5th	
	Figurate numbers of the 5th order		6th	
	Figurate numbers of the 6th order		7th	

Therefore the figurate numbers

of the	1st order	are	1 . 1 . 1 . 1 . 1 . &c.
	2d order		1 . 2 . 3 . 4 . 5 . &c.
	3d order		1 . 3 . 6 . 10 . 15 . &c.
	4th order		1 . 4 . 10 . 20 . 35 . &c.
	5th order		1 . 5 . 15 . 35 . 70 . &c.

Hence it is manifest, that, to find a general expression for a figurate number of any order, is the same thing as to find the sum of all the figurate numbers of the preceding order, so far. Let n be put to denote the distance of any such number from the beginning of its respective order, or the number of terms in the preceding order whereof it is composed: then it is evident, by inspection, that the sum of the first order, or the nth term of the second, will be truly expressed by n, the number of terms from the beginning. It is also evident, from sect. 14, p. 203, that the sum of the second order, $1 + 2 + 3 + 4 \ldots\ldots n$, will be $\frac{n^2}{2} + \frac{n}{2}$ $(= \frac{n}{1} \times \frac{n+1}{2})$ which, according to the preceding observation, is also the value of the nth term of the third order. Hence, if the numbers, 1, 2, 3, 4, 5, *&c.* be successively written instead of n, in the general expression $\frac{n^2}{2} + \frac{n}{2}$, we shall thence have $\frac{1}{2} + \frac{1}{2}$, $\frac{4}{2} + \frac{2}{2}$, $\frac{9}{2} + \frac{3}{2}$, $\frac{16}{2} + \frac{4}{2}$, $\frac{25}{2} + \frac{5}{2}$, *&c.* for the values of the first, second, third, fourth, fifth, *&c.* terms of this order, respectively; whence it appears, that the series $1 + 3 + 6 + 10 + 15 + 21$, *&c.* may be resolved into these two others, *viz.*

$\frac{1}{2} + \frac{4}{2} + \frac{9}{2} + \frac{6}{2} + \frac{25}{2} + \frac{36}{2}$, &c. and
$\frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \frac{4}{2} + \frac{5}{2} + \frac{6}{2}$, &c.

The former of which being a series of squares, its sum will therefore be $= \frac{n^3}{6} + \frac{n^2}{4} + \frac{n}{12}$ (*by case* 2, p. 203), and that of the latter series (*by case* 1, p. 203) appears to be $\frac{n^2}{4} + \frac{n}{4}$: and the aggregate of both, which is $\frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3}$ (or $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$) will be the true value of the proposed series $1 + 3 + 6 + 10 + 15$ *&c.* continued to n terms, and therefore equal, likewise, to the nth term of the next superior order, $1 + 4 + 10 + 20 + 35$, *&c.* Let, therefore, 1, 2, 3, 4, 5, *&c.* (as above) be successively written for n in this general expression, $\frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3}$, and it will become $\frac{1}{6} + \frac{1}{2} + \frac{1}{3}$, $\frac{8}{6} + \frac{4}{2} + \frac{2}{3}$, $+ \frac{27}{6} + \frac{9}{2} + \frac{3}{3}$, $\frac{64}{6} + \frac{16}{2} + \frac{4}{3}$, *&c.* for the values of the first, second, third, fourth, *&c.*

terms of the fourth order, respectively; whence it appears that the series $1 + 4 + 10 + 20 + 35$, &c. may be resolved into these three others, *viz.*

$$\frac{1 + 8 + 27 + 64 + 125 + 216 \ldots . n^3}{6},$$

$$\frac{1 + 4 + 9 + 16 + 25 + 36 \ldots . n^2}{2},$$

$$\frac{1 + 2 + 3 + 4 + 5 + 6 \ldots . n}{3}:$$

whereof the sums are $\frac{n^4}{24} + \frac{n^3}{12} + \frac{n^2}{24}$, $\frac{n^3}{6} + \frac{n^2}{4} + \frac{n}{12}$, and $\frac{n^2}{6} + \frac{n}{6}$ (*by* p. 202 and 203) the aggregate of which, or $\frac{n^4}{24} + \frac{n^3}{4} + \frac{11n^2}{24} + \frac{n}{4}$ $\left(= \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}\right)$ will consequently be the true value of the whole series. After the same manner the sum of the fifth order will appear to be $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5}$; from whence the law of continuation is manifest. And it may not be amiss to observe here, that though the conclusions thus brought out are derived by means of the sums of powers determined in the preceding section, yet the same values may be otherwise obtained, by a direct investigation, from either of the two general methods there laid down.

In order now to find the sum of the reciprocals of any series of figurate numbers; suppose $1 + b + bc + bcd + bcde + bcdef +$ &c. to be a series whose terms continually decrease, from the first to the last, so that the last may vanish, or become indefinitely small: then, by taking the excess of every term above the next following one, we shall have $1 - b$, $b \times \overline{1 - c}$, $bc \times \overline{1 - d}$, $bcd \times \overline{1 - e}$, $bcde \times \overline{1 - f}$, &c. The sum of all which is evidently equal to the excess of the first term above the last, or equal to the first term, *barely;* because the last is supposed to vanish, or to be indefinitely small in respect of the first. Hence it appears, that $\overline{1-b} + b \times \overline{1-c} + bc \times \overline{1-d} + bcd \times \overline{1-e} + bcde \times \overline{1-f}$, &c. $= 1$.

Let b be now taken $= \frac{m}{a}$, $c = \frac{m+p}{a+p}$, $d = \frac{m+q}{a+q}$, $e = \frac{m+r}{a+r}$, $f = \frac{m+s}{a+s}$, &c. Then, $1-b$ being $= \frac{a-m}{a}$, $1-c = \frac{a-m}{a+p}$, $1-d = \frac{a-m}{a+q}$, $1-e = \frac{a-m}{a+r}$, &c. we shall, by substituting these several values in the above equation, have $\frac{a-m}{a} + \frac{m}{a} \times \frac{a-m}{a+p} + \frac{m}{a} \times \frac{m+p}{a+p} \times \frac{a-m}{a+q} + \frac{m}{a} \times \frac{m+p}{a+p} \times \frac{m+q}{a+q} \times \frac{a-m}{a+r} +$ &c. $= 1$; and consequently $1 + \frac{m}{a+p} + \frac{m}{a+p} \times \frac{m+p}{a+q} + \frac{m}{a+p} \times \frac{m+p}{a+q} \times \frac{m+q}{a+r} +$ &c. $= \frac{a}{a-m}$; by dividing the whole by $\frac{a-m}{a}$.

Hence, if q be taken $= 2p$, $r = 3p$, $s = 4p$, &c. and β be put $= a + p$, we shall have $1 + \frac{m}{\beta} + \frac{m \cdot m+p}{\beta \cdot \beta+p} + \frac{m \cdot m+p \cdot m+2p}{\beta \cdot \beta+p \cdot \beta+2p} + \frac{m \cdot m+p \cdot m+2p \cdot m+3p}{\beta \cdot \beta+p \cdot \beta+2p \cdot \beta+3p}$ + &c. *ad infinitum*, $= \frac{\beta-p}{\beta-p-m}$; which, when $p = 1$, becomes $1 + \frac{m}{\beta} + \frac{m \cdot m+1}{\beta \cdot \beta+1} + \frac{m \cdot m+1 \cdot m+2}{\beta \cdot \beta+1 \cdot \beta+2}$ + &c. $= \frac{\beta-1}{\beta-m-1}$: this, by taking $m = 1$, and $\beta = n$, gives $1 + \frac{1}{n} + \frac{1 \cdot 2}{n \cdot \overline{n+1}} + \frac{1 \cdot 2 \cdot 3}{n \cdot \overline{n+1} \cdot \overline{n+2}} +$

$+ \frac{1 . 2 . 3 . 4}{n . \overline{n+1} . \overline{n+2} . \overline{n+3}} + \&c. = \frac{n-1}{n-2}$; exhibiting the general value of a series of the reciprocals of figurate numbers, infinitely continued; whereof the order is represented by n: from whence as many particular values as you please may be determined. Thus, by expounding n by 3, 4, 5, *&c.* successively, it appears that

$$1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15}, \&c. = 2,$$

$$1 + \frac{1}{4} + \frac{1}{10} + \frac{1}{20} + \frac{1}{35}, \&c. = \frac{3}{2},$$

$$1 + \frac{1}{5} + \frac{1}{15} + \frac{1}{35} + \frac{1}{70}, \&c. = \frac{4}{3},$$

and so on, for any higher order; but the sums of the two first, or lowest orders, cannot be determined, these being infinite.

By interpreting β and m by different values, the sums of various other series may be deduced from the same general equation. Thus, in the first place, let $\beta = m + 2$; so shall the said equation become $1 + \frac{m}{m+2} + \frac{m . \overline{m+1}}{\overline{m+2} . \overline{m+3}} + \frac{m . \overline{m+1}}{\overline{m+3} . \overline{m+4}} + \frac{m . \overline{m+1}}{\overline{m+4} . \overline{m+5}}, \&c. = m + 1$; which, divided by $m . \overline{m+1}$, gives

$$\frac{1}{m . \overline{m+1}} + \frac{1}{\overline{m+1} . \overline{m+2}} + \frac{1}{\overline{m+2} . \overline{m+3}} + \frac{1}{\overline{m+3} . \overline{m+4}} \&c. = \frac{1}{m}.$$

Again, by taking $\beta = m + 3$, and dividing the whole equation by $m . \overline{m+1} . \overline{m+2}$, we have

$$\frac{1}{m . \overline{m+1} . \overline{m+2}} + \frac{1}{\overline{m+1} . \overline{m+2} . \overline{m+3}} + \frac{1}{\overline{m+2} . \overline{m+3} . \overline{m+4}}, \&c. = \frac{1}{m . \overline{m+1} . 2}.$$

In like manner we shall have $\frac{1}{m\cdot\overline{m+1}\cdot\overline{m+2}\cdot\overline{m+3}} + \frac{1}{\overline{m+1}\cdot\overline{m+2}\cdot\overline{m+3}\cdot\overline{m+4}}$, &c. $= \frac{1}{m\cdot\overline{m+1}\cdot\overline{m+2}\cdot 3}$. From whence the law for continuing the sums of these last kinds of series is manifest; by which it appears, that if, instead of the last factor, in the denominator of the first term, the excess thereof above the first factor be substituted, the fraction thence arising will truly express the value of the whole infinite series.

A few other particular cases will further show the use of the general equations above exhibited.

Let the sum of the series $1 + \frac{2}{5} + \frac{2}{5} \times \frac{4}{7} + \frac{2}{5} \times \frac{4}{7} \times \frac{6}{9} + \frac{2}{5} \times \frac{4}{7} \times \frac{6}{9} \times \frac{8}{11}$, &c. *ad infinitum*, be required.

Here, by comparing the proposed series with $1 + \frac{m}{\beta} + \frac{m\cdot\overline{m+p}}{\beta\cdot\overline{\beta+p}}$ + &c. $(= \frac{\beta-p}{\beta-p-m})$, we have $m = 2$, $\beta = 5$, and $p = 2$; and consequently $\frac{\beta-p}{\beta-p-m} = 3 =$ the true value of the series.

Let the sum of an infinite series of this form, *viz.* $\frac{1}{1\cdot2\cdot3,\text{ \&c.}} + \frac{1}{2\cdot3\cdot4,\text{ \&c.}} + \frac{1}{3\cdot4\cdot5,\text{ \&c.}}$ + &c. be demanded.

Here (according to the preceding rule) we have

$$\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4}, \text{ \&c.} = \frac{1}{1\cdot1} = 1;$$

$$\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5}, \text{ \&c.} = \frac{1}{1\cdot2\cdot2} = \frac{1}{4};$$

$$\frac{1}{1\cdot2\cdot3\cdot4} + \frac{1}{2\cdot3\cdot4\cdot5} + \frac{1}{3\cdot4\cdot5\cdot6}, \text{ \&c.} = \frac{1}{1\cdot2\cdot3\cdot3} = \frac{1}{18};$$

&c. &c.

If, instead of the whole infinite series, you want the sum of a given number of the leading terms only; then let the value of the remaining part be found, as above, and subtracted from the whole, and you will have your desire.

Thus, for instance, let it be required to find the sum of the ten first terms of the series $\frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\frac{1}{4\,.\,5}$, &c. Then the remaining part, $\frac{1}{11\,.\,12}+\frac{1}{12\,.\,13}+\frac{1}{13\,.\,14}+\frac{1}{14\,.\,15}$, &c. being $=\frac{1}{11}$ (*by the rule above*), and the whole series $=1$, the value here sought will therefore be $1-\frac{1}{11}=\frac{10}{11}$. The like of others.

The sums of series arising from the multiplication of the terms of a rank of figurate numbers into those of a decreasing geometrical progression, are deduced in the following manner.

By the theorem for involving a binomial (given at p. 40, and demonstrated hereafter) it is known that $\frac{1}{\overline{1-x}\,|^{m}}$ (or $\overline{1-x}\,|^{-m}$) is $=1+mx+m\,.\,\frac{m+1}{2}\,.\,x^2+m\,.\,\frac{m+1}{2}\,.\,\frac{m+2}{3}\,.\,x^3+m\,.\,\frac{m+1}{2}\,.\,\frac{m+2}{3}\,.\,\frac{m+3}{4}\,.\,x^4$, &c. In which equation let m be expounded by 1, 2, 3, 4, 5, &c. successively, so shall

1°. $\frac{1}{1-x}=1+x+x^2+x^3+x^4+x^5+$ &c.

2°. $\frac{1}{\overline{1-x}\,|^{2}}=1+2x+3x^2+4x^3+5x^4+6x^5$, &c.

3°. $\frac{1}{\overline{1-x}\,|^{3}}=1+3x+6x^2+10x^3+15x^4+21x^5$, &c.

4°. $\frac{1}{\overline{1-x}\,|^{4}}=1+4x+10x^2+20x^3+35x^4+56x^5$, &c.

5°. $\frac{1}{\overline{1-x}|^{5}} = 1 + 5x + 15x^2 + 35x^3 + 70x^4 + 126x^5$, &c.

6°. $\frac{1}{\overline{1-x}|^{6}} = 1 + 6x + 21x^2 + 56x^3 + 126x^4 + 252x^5$, &c.

All which series (whereof the sums are thus given) are ranks of the different orders of figurate numbers, multiplied by the terms of the geometrical progression 1, x, x^2, x^3, x^4, &c.

From these equations the sums of series composed of the terms of a rank of powers, drawn into those of a geometrical progression, such as $1 + 4x + 9x^2 + 16x^3$, &c. and $1 + 8x + 27x^2 + 64x^3$, &c. may also be derived; there being, as appears from the former part of this section, a certain relation between the terms of a series of powers and those of figurate numbers; the latter being there determined by means of the former. To find here the converse relation, or to determine the former from the latter, it will be expedient to multiply the several equations above brought out, by a certain number of terms of an assumed series $1 + Ax + Bx^2 + Cx^3$, &c. in order that the coefficients of the powers of x may, by regulating the values A, B, C, D, *&c.* become the same as in the series given.

Thus, if the series given be $1 + 4x + 9x^2 + 16x^3 + 25x^4$, &c.; then, by multiplying our third equation by $1 + Ax$, we shall have $\frac{1 + Ax}{\overline{1-x}|^{3}} = 1 + \overline{3 + A} \times x^2 + \overline{6 + 3A} \times x^2 + \overline{10 + 6A} \times x^3 +$ &c. which series, it is evident, by inspection, will be exactly the same, in every term, with the proposed one, if the quantity A be taken $= 1$. The sum of the said series, infinitely continued, is therefore truly represented by $\frac{1 + x}{\overline{1-x}|^{3}}$.

In like manner, if the fourth equation $\frac{1}{\overline{1-x}|^{4}} = 1 + 4x + 10x^2 + 20x^3 + 35x^4$, &c. be multiplied by

$1 + Ax + Bx^2$, there will arise $\frac{1 + Ax + Bx^2}{\overline{1 - x}|^4} = 1 + \overline{4 + A} \times x + \overline{10 + 4A + B} \times x^2 + \overline{20 + 10A + 4B} \times x^3$, &c. where, the several terms of the series being compared with those of the series $1 + 8x + 27x^2 + 64x^3$, &c. we have $4 + A = 8$, and $10 + 4A + B = 27$; whence $A = 4$, and $B = 1$; and consequently, by substituting these values, $\frac{1 + 4x + x^2}{\overline{1 - x}|^4} = 1 + 8x + 27x^2 + 64x^3 + 125x^4$, &c.

Again, by multiplying the fifth equation, $\frac{1}{\overline{1 - x}|^5} = 1 + 5x + 15x^2 + 35x^3$, &c. by $1 + Ax + Bx^2 + Cx^3$, it becomes $\frac{1 + Ax + Bx^2 + Cx^3}{\overline{1 - x}|^5} = 1 + \overline{5 + A} \times x + \overline{15 + 5A + B} \times x^2 + \overline{35 + 15A + 5B + C} \times x^3$, &c. And, by comparing the several terms of the series with those of $1 + 16x + 81x^2 + 256x^3$, &c. we get $5 + A = 16$, $15 + 5A + B = 81$, and $35 + 15A + 5B + C = 256$: whence $A = 11$, $B\ (= 81 - 15 - 55) = 11$, and $C\ (= 256 - 35 - 220) = 1$; and consequently $\frac{1 + 11x + 11x^2 + x^3}{\overline{1 - x}|^5} = 1 + 16x + 81x^2 + 256x^3$, &c.

By proceeding the same way, it will be found, that $\frac{1 + 26x + 66x^2 + 26x^3 + x^4}{\overline{1 - x}|^6} = 1 + 2^5x + 3^5x^2 + 4^5x^3 +$ &c. &c.

And, *universally*, putting $a = m$, $b = m \cdot \frac{m + 1}{2}$, $c = m \cdot \frac{m + 1}{2} \cdot \frac{m + 2}{3}$, &c. and multiplying the general equation $\frac{1}{\overline{1 - x}|^m} = 1 + ax + bx^2 + cx^3 + dx^4$, &c. by $1 + Ax + Bx^2 + Cx^3 + Dx^4$, &c. there arises $\frac{1 + Ax + Bx^2, \&c.}{\overline{1 - x}|^m} = 1 + \overline{a + A} \times x +$

$\overline{b + aA + B} \times x^2 + \overline{c + bA + aB + C} \times x^3$, &c. The terms of which series being compared with those of the series $1 + 2^n x + 3^n x^2 + 4^n x^3 + 5^n x^4$, &c. we have $A = 2^n - a$, $B = 3^n - aA - b$, $C = 4^n - aB - bA - c$, $D = 5^n - aC - bB - cA - d$, &c. where the law of continuation is manifest; and where, from the law observed in all the preceding cases, it appears, that the value of m must exceed the index n, of the given series of powers, by a unit; and that the series $1 + Ax + Bx^2 + Cx^3$, &c. will always consist of n terms; whereof the coefficients of the first and last, the second and last but one, *&c.* will be respectively equal to each other: so that, having found from the preceding equations as many of the quantities A, B, C, *&c.* as are expressed by $\frac{1}{2}n - 1$, the others will be given from thence, and, consequently, $\frac{1 + Ax + Bx^2 + Cx^3, \&c.}{\overline{1 - x}|^{n+1}}$ the true value of the proposed series $1 + 2^n x + 3^n x^2 + 4^n x^3$, &c. Thus, for example, let $n = 6$: then $m = 7 = a$, $b = 28$, $A = 64 - 7 = 57$, $B = 729 - 399 - 28 = 302$; and therefore

$$\frac{1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5}{\overline{1 - x}|^7} = 1 + 2^6 x + 3^6 x^2 + 4^6 x^3, \&c.$$

and so of others.

These equations, or theorems, give the sum of the whole series, infinitely continued; but from thence the sum of any assigned number of terms may be determined, not only when the coefficients are a series of powers, but likewise when they are produced by factors that are unequal: the method of which I shall instance in finding the sum of t terms of the series $\overline{f - p} \cdot \overline{g - q} \cdot z^r + \overline{f - 2p} \cdot \overline{g - 2q} \cdot z^{r+v} + \overline{f - 3p} \cdot \overline{g - 3q} \cdot z^{r+2v}$ + &c. Which series, by actually multiplying the factors together, is resolved into the three following ones.

$$fgz^r \times \overline{1 + z^v + z^{2v} + z^{3v} + z^{4v}}, \text{\&c.}$$
$$-\overline{fq + gp} \cdot z^r \times \overline{1 + 2z^v + 3z^{2v} + 4z^{3v}}, \text{\&c.}$$
$$+pqz^r \times \overline{1 + 4z^v + 9z^{2v} + 16z^{3v}}, \text{\&c.}$$

The sum of the first of these, infinitely continued, supposing $x = z^v$, will be $= \frac{fgz^r}{1 - x}$; that of the second $= -\frac{\overline{fq + gp} \cdot z^r}{\overline{1 - x}|^2}$; and that of the third $= \frac{\overline{1 + x} \cdot pqz^r}{\overline{1 - x}|^3}$, *by what has been above determined;* and consequently the sum of all the three equal to $\frac{z^r}{1 - x} \times fg - \frac{fq + gp}{1 - x} + \frac{pq \cdot \overline{1 + x}}{\overline{1 - x}|^2}$ = the whole infinite series $\overline{f - p} \cdot \overline{g - q} \cdot z^r + \overline{f - 2p} \cdot \overline{g - 2q} \cdot z^{r+v} +$ *&c.* But the sum of the t first terms only is wanted; therefore the sum of all the remaining terms, after the t first, must be found in like manner, and be deducted from the sum of the whole, here given. Now, to do this, we are first to get the leading term of the said remaining ones; which, according to the law of the series, will be expressed by $\overline{f - p - tp} \cdot \overline{g - q - tq} \cdot z^{r+tv}$: whence, if we make $f - tp = h$, $g - tq = k$, and $r + tv = s$, it is evident, that the series to be deducted will be $\overline{h - p} \cdot \overline{k - q} \cdot z^s + \overline{h - 2p} \cdot \overline{k - 2q} \cdot z^{s+v}$, &c. which, having the very same form with that first proposed, its sum will therefore be had by barely writing h for f, k for g, and s for r, in the value above determined: which, thereby, becomes

$$\frac{z^s}{1 - x} \times \overline{hk - \frac{hq + kp}{1 - x} + \frac{pq \cdot \overline{1 + x}}{\overline{1 - x}|^2}}.$$

In the same manner, supposing the t first terms of the series $\overline{a-p}\cdot\overline{b-p}\cdot\overline{c-p}\cdot\overline{d-p}\cdot$ &c. $\times z^r+\overline{a-2p}\cdot\overline{b-2p}\cdot\overline{c-2p}\cdot\overline{d-2p}\cdot$ &c. $\times z^{r+v}$, &c. were to be required; by putting the continual product of all the quantities a, b, c, d, &c. $=$ P; the sum of all the products $(\frac{P}{a}+\frac{P}{b}+\frac{P}{c}$, &c.) that arise by omitting one letter in each, $=$ Q; the sum of all those $(\frac{P}{ab}+\frac{P}{ac}$, &c.), by omitting two letters, $=$ R, &c. we shall here have

$$\frac{z^r}{1-x}\times\left\{\begin{array}{l}\overline{P-\frac{Qp}{1-x}+\frac{Rp^2\cdot\overline{1+x}}{\overline{1-x}\,\big|^2}} \\ \frac{Sp^3\cdot\overline{1+4x+x^2}}{\overline{1-x}\,\big|^3}+\frac{Tp^4\cdot\overline{1+11x+11x^2+x^3}}{\overline{1-x}\,\big|^4},\end{array}\right.$$

&c. for the sum of the whole infinite series: and if we make $\acute{a}=a-tp$, $\acute{b}=b-tp$, $\acute{r}=r+tv$, &c. it is evident that the sum of the remaining terms, after the t first, will be truly expressed by

$$\frac{z^{\acute{r}}}{1-x}\times\overline{\acute{P}-\frac{\acute{Q}p}{1-x}+\frac{\acute{R}p^2\cdot\overline{1+x}}{\overline{1-x}\,\big|^2}-\frac{\acute{S}p^3\cdot\overline{1+4x+x^2}}{\overline{1-x}\,\big|^3}},$$

&c. where $x=z^v$, and $\acute{P}$, $\acute{Q}$, $\acute{R}$, $\acute{S}$, &c. are the same in relation to $\acute{a}$, $\acute{b}$, $\acute{c}$, $\acute{d}$, &c. as P, Q, R, S, &c. in respect to a b, c, d, &c.

A multitude of other cases and examples might be given, there not being, in the whole scope of the mathematical sciences, a subject of greater variety and intricacy than this business of series: but to pursue it farther here would be inconsistent with the general plan of this work. Such, therefore, as are desirous of a greater insight into the matter, may, if they please, turn to my *Miscellanies*, where it is carried to a greater length.

From the series for figurate numbers, derived in the former part of this section, the investigation of a general theorem for determining how many different combinations any number of things will admit of, when taken two by two, three by three, *&c.* may be very easily deduced. *Let the number of things in each combination be, first, supposed two only; and let n be, universally, put to represent the whole number of things or letters, a, b, c, d,* &c. *to be combined.* When the number of things is only two, as a and b, it is evident that there can be only one combination (ab); but, if n be increased by 1, or the letters to be combined be three, as a, b, c, then it is plain that the number of combinations will be increased by 2, the number of the preceding letters a and b; since, with each of those, the new letter c may be joined; and therefore the whole number of combinations, in this case, will be truly expressed by $1 + 2$. Again, if n be increased by one more, or the whole number of letters be four, as a, b, c, d; then it will appear that the number of combinations must be increased by 3, since 3 is the number of the preceding letters, with which the new letter d can be combined, and therefore will here be truly expounded, by $1 + 2 + 3$. And, by reasoning in the same manner, it will appear, that the whole number of combinations of two, in five things, will be $1 + 2 + 3 + 4$; in six things, $1 + 2 + 3 + 4 + 5$; and in seven, $1 + 2 + 3 + 4 + 5 + 6$, *&c.* Whence, universally, the number of combinations of n things, taken two by two, is $= 1 + 2 + 3 + 4 + \ldots\ldots \overline{n - 1}$: which being a series of figurate numbers of the second order, where the number of terms is $n - 1$, the sum thereof, by case 1, p. 203, will therefore be truly defined by $\frac{n-1}{1} \times \frac{n}{2}$, or $n \times \frac{n-1}{2}$.

Let now the number of quantities in each combination be supposed to be three.

It is plain, that, in three things, a, b, c, there can be only one combination; but, if n be increased by 1, or the number of things be 4, as a, b, c, d, then will the number of combinations be increased by (3) the number of all the combinations of two, in the preceding letters a, b, c; since with each two of those the new letter d may be combined; therefore the number of combinations, in this case, is $1 + 3$. Again, if n be supposed to be increased by 1 more, or the number of letters to become five, as a, b, c, d, e; then the number of combinations will be increased by six more ($= 1 + 2 + 3$), that is, by all the combinations of two, in the four preceding letters, a, b, c, d: since, as before, with each two of those, the new letter e may be combined. Hence the number of combinations of n things, taken three by three, appears to be $1 + 3 + 6 + 10$, &c. continued be $n - 2$ terms; which being a series of figurate numbers of the third order, the value thereof, by what is before determined (p. 214) will be truly expressed by $\frac{n-2}{1} \times \frac{n-1}{2} \times \frac{n}{3}$, or its equal, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$.

And universally, since it appears that increasing the number of letters by 1 always increases the number of combinations by all the combinations of the next inferior order with the preceding letters (for this obvious reason, that to each of these last combinations, the new letter may be joined), it is manifest, that the combinations, of any order, observe the same law, and are generated in the very same manner as figurate numbers, and therefore may be exhibited by the same general expressions; only, as there are 2, 3, 4, 5, &c. things necessary to form the first, or one single combination, according to the different cases, it is plain that the number of terms must be less by 1, 2, 3, &c. respectively, than (n) the number of things; and, therefore, instead of n, in the aforesaid general expressions, we must substitute $n - 1$, $n - 2$, or $n - 3$, &c. respectively, to have the true value here. Hence, the number of combinations of two things, in n things, will be $\frac{n-1}{1} \times \frac{n}{2}$, or $\frac{n}{1} \times \frac{n-1}{2}$; of three, $\frac{n-2}{1} \times \frac{n-1}{2} \times \frac{n}{3}$, or $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$; of four, $\frac{n-3}{1}\times\frac{n-2}{2}\times\frac{n-1}{3}\times\frac{n}{4}$, or $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$ (*vid. p.* 215): whence, universally, the number of combinations in the number, n, of things, taken two by two, three by three, &c. will be expressed by $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$, &c. continued to as many factors as there are things in each combination.

From this last general expression, showing the combinations which any number of quantities will admit of, the known theorem for raising a binomial to any given power, is very easily and naturally derived.

For it is plain that $aa\left.\begin{matrix}+b\\+c\end{matrix}\right\}a+bc=\overline{a+b}\times\overline{a+c}$; which, multiplied by $a+d$, gives $a^3\left.\begin{matrix}+b\\+c\\+d\end{matrix}\right\}a^2+\left.\begin{matrix}bc\\bd\\cd\end{matrix}\right\}a+bcd=\overline{a+b}\times\overline{a+c}\times\overline{a+d}$; and this, again, multiplied by $a+e$, gives

$$a^4+\left.\begin{matrix}+b\\+c\\+d\\+e\end{matrix}\right\}a^3+\left.\begin{matrix}+bc\\+bd\\+be\\+cd\\+ce\\+de\end{matrix}\right\}a^2+\left.\begin{matrix}+bcd\\+bce\\+bde\\+cde\end{matrix}\right\}a+bcde=\overline{a+b}\times\overline{a+c}\times\overline{a+d}\times\overline{a+e}.$$

Whence it appears, that the coefficient of a, in the second term, is always the sum of all the other quantities b, c, d, &c. added together; and that the coefficient of the third term is the sum of all the products of those quantities, or of all their possible combinations, taken two by two; since, from the nature of multiplication, they must be all concerned alike, in every term: whence it is also manifest, that the coefficient of the fourth term must be the sum of all the solids of the same quantities, or of all their possible combinations, taken three by three, &c. &c.

Hence, if the number of the quantities b, c, d, e, &c. or the number of the factors, $a + b$, $a + c$, $a + d$, $a + e$, &c. to be multiplied continually together, be denoted by n, it follows, that the number of letters, or quantities, in the coefficient of the second term of the product will likewise be denoted by n; that the number of all their products, or of all the combinations of two, in the coefficient of the third term, will be $n \times \frac{n-1}{2}$ (it having been shown above that the number of combinations of n things, taken two by two, is $n \times \frac{n-1}{2}$); and that the number of all the solids of those quantities, or all the combinations of three, in the coefficient of the fourth term, will be $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, &c. Therefore, if all the quantities b, c, d, e, &c. be now taken equal to each other, it is evident, that $\overline{a+b} \times \overline{a+c} \times \overline{a+d} \times \overline{a+e}$, &c. will become $\overline{a+b} \times \overline{a+b} \times \overline{a+b} \times \overline{a+b}$, &c. or $\overline{a+b}|^{n}$; and that the coefficient of the power of a, in the second term of the product, will be nb; in the third $n \times \frac{n-1}{2} b^2$ (since all the rectangles, as well as all the solids, &c. do here become equal); and in the fourth $n \times \frac{n-1}{2} \times \frac{n-2}{3} b^3$, &c. But it is evident, from the nature of multiplication, that the powers of a, in the second, third, fourth, &c. terms of $a + b$ raised to the power n, are a^{n-1}, a^{n-2}, a^{n-3}, &c. Therefore $\overline{a+b}^{n}$, or $a + b$ raised to the power n, is truly expressed by $a^n + nba^{n-1} + n \times \frac{n-1}{2} b^2 a^{n-2} + n \times \frac{n-1}{2} \times \frac{n-2}{3} b^3 a^{n-3}$, &c. or $a^n + na^{n-1} b + n \times \frac{n-1}{2} a^{n-2} b^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} b^3$, &c. as was to be shown.

SECTION XVI.

Of Interest and Annuities.

INTEREST may be either simple or compound: simple interest is that which is paid for the loan of any principal, or sum of money; lent out for some limited time, at a certain rate *per cent.* agreed upon between the borrower and the lender, and is always proportional to the time. Thus, if the rate agreed upon be 4 *per cent. per ann.* or, which is the same thing, if the interest of 100*l.* for one year be 4*l.* then the simple interest of the same sum for two years will be 8*l.*; for three years 12*l.*; and for four years 16*l.*; and so on for any other time in proportion.

Compound interest is that which arises by leaving the simple interest of any sum of money, after it becomes due, together with the principal, in the hands of the borrower, and thereby converting the whole into a new principal. Thus, he who lets out 100*l.* for one year, at the rate of 4 *per cent.* has a right to receive 104*l.* at the year's end; which sum he may leave in the borrower's hands, a second year, as a new principal, in order to receive interest for the whole; and this interest (which will be found 4*l.* 3*s.* $2\frac{1}{2}$*d.*), together with 4*l.* the interest of the first principal, for the first year, will be the compound interest of 100*l.* for two years; and so on, for any greater number of years. But I shall first give the investigation of the theorems for simple interest.

Let the rate *per cent.* or the interest of 100*l.* for one year $= r$; the months, weeks, or days in one year $= t$; the months, weeks, or days which any sum, a, is lent out for $= n$; and the amount of that sum, in the said time, *viz.* principal and interest, $= b$.

Then it will be, as 100 is to r (the interest of 100*l.*) so is the proposed sum (a) to $\frac{ar}{100}$, the interest of that sum, for the same time. Again, as t, the time in which the said interest is produced, is to n (the time proposed), so is $\frac{ar}{100}$, the interest in the former of these times, to $\frac{anr}{100t}$, that in the latter; which, added to a, the principal, gives $a + \frac{anr}{100t} = b$, the whole amount: from whence we also have $a = \frac{100bt}{100t + nr}$, $r = \frac{100t \times \overline{b - a}}{an}$, and $n = \frac{100t \times \overline{b - a}}{ar}$: the use of which equations, or theorems, will appear by the following examples:

Examp. 1. What is the amount of 550*l.* at 4 *per cent.* in seven months?

In this case we have $a = 550$, $r = 4$, $t = 12$, $n = 7$: and consequently $a + \frac{anr}{100t} = 550 + \frac{550 \cdot 7 \times 4}{100 \times 12} = 562\frac{5}{6}l.$ or 562*l.* 16*s.* 8*d.* the true value sought.

Examp. 2. What is the interest of 1*l.* for one day, at the rate of 5 *per cent.*?

Here r being $= 5$, $t = 365$, $a = 1$, and $n = 1$, we have $\frac{anr}{100t} = \frac{5}{100 \times 365} = \frac{1}{100 \times 73} = 0.0001369863$, &c. = the decimal parts of a pound required.

Examp. 3. What sum, in ready money, is equivalent to 600*l.* due 9 months hence, allowing 5 *per cent.* discount?

Here r being $= 5$, $t = 12$, $n = 9$, and $b = 600$, we have a (*by theorem* 2) $= \frac{100 \times 600 \times 12}{100 \times 12 + 9 \times 5} = 578{,}313l.$ or 578*l.* 6*s.* $3\frac{1}{4}d.$ which is the value required.

Examp. 4. At what rate of interest will 300*l.* in fifteen months amount to, or raise a stock of, 330*l.*?

In this case, we have given $t = 12$, $n = 15$, $a = 300$, and $b = 330$; whence (*by theorem* 3) r will come out $= \frac{100 \times 12 \times 30}{300 \times 15} = 8$; therefore 8 *per cent.* is the rate required.

Examp. 5. In how many days will 365*l.* at the rate of 4 *per cent.* amount to, or raise a stock of, 400*l.*?

Here (*by theorem* 4) we have $n = \frac{100 \times 365 \times 35}{365 \times 4}$ $= 875 =$ the number of days required.

Of Annuities or Pensions in Arrear, computed at Simple Interest.

Annuities or pensions in arrear are such as, being payable, or becoming due, yearly, remain unpaid any number of years: and we are to compute what all those payments will amount to, allowing simple interest for their forbearance, from the time each particular payment becomes due: in order to which,

Let $\begin{cases} A = \text{the annuity, pension, or yearly rent.} \\ n = \text{the time, or number of years, it is forborne.} \\ r = \text{the interest of } 1l. \text{ for one year.} \\ m = \text{the amount of the annuity and its interest.} \end{cases}$

Then, as $1 : r :: A : rA$, the interest of the proposed sum or pension A, for one year; which, as the last year's rent but one is forborne only one year, will express the whole interest of that rent, or payment: moreover, since the last year's rent but two is forborne two years, its interest will be $2rA$: and, in the same manner, that of the last year's rent but three, will appear to be $3rA$, &c. &c. whence it is manifest that the sum total of all these, or the whole interest to be received at the expiration of n years, for the forbearance of the proposed annuity or pension, will be truly defined by the arithmetical progression $rA + 2rA + 3rA + 4rA + 5rA$, &c. continued to $n - 1$ terms, that is, to as many terms as there are years, excepting the last. But the sum of this progression is equal to $n \times \frac{n-1}{2} \times rA$ (*by theor.* 4, *sect.* 10). Therefore, if to this the aggregate of all the rents, or nA, be added, we shall have $nA + \frac{n \times \overline{n-1}}{2} \times rA = m$:

whence we also have $A = \frac{m}{n + n \times \frac{n-1}{2} \times r}$, $r = \frac{2m - 2nA}{n \times \overline{n-1} \times A}$, and $n = \sqrt{\frac{2m}{rA} + p^2} - p$; supposing $p = \frac{1}{r} - \frac{1}{2}$.

Examp. 1. If 600*l.* yearly rent, or pension, be forborne five years, what will it amount to, allowing 4 *per cent.* interest for each payment from the time it becomes due?

Here we have given $A = 600$, $n = 5$, and $r = .04$ (for as 100 : 4 : : 1 : .04), which values substituted, in *theorem* 1, give $m = (nA + n \times \frac{n-1}{2} Ar = 3000 + 240) = 3240$*l.* for the value that was to be found.

Examp. 2. What annuity, or yearly pension, being forborne five years, will, in that time, amount to, or raise a stock of, 3240*l.* at 4 *per cent.* interest?

In this case we have given $n = 5$, $r = .04$, and $m = 3240$, and therefore, by *theorem* 2, $A \left(= \frac{m}{n + \frac{1}{2}n \times \overline{n-1} \times r} = \frac{3240}{5+4}\right) = 600$; which is the annuity required.

Examp. 3. At what rate of interest will an annuity of 560*l.* in seven years, raise a stock of 4508*l.*?

In this case we have given $A = 560$, $n = 7$, and $m = 4508$; whence (by *theor.* 3) we have $r \left(= \frac{2m - 2nA}{n \times \overline{n-1} \times A}\right) = \frac{9016 - 7840}{42 \times 560} = .05$ = the interest of 1*l.* for one year; therefore it will be, as 1 : ,05 : : 100 : 5 *per cent.* the rate required.

Examp. 4. How long must an annuity of 560*l.* be forborne, to raise a stock of 4508*l.* supposing interest to be 5 *per cent.*?

Here, we have given $A = 560$, $r = .05$, $m = 4508$; whence, *by theorem* 4, we also have $p \left(= \frac{1}{r} - \frac{1}{2}\right) = 19.5$; and consequently $n \left(= \sqrt{\frac{2m}{rA} + p^2} - p\right) = 7$; which is the number of years required.

Note. If the rent or pension be payable half-yearly, or quarterly, the method of proceeding will be still the same, provided n be always taken to express the number of payments, and r the interest of 1*l.* for the time in which the first payment becomes due. Thus, if it were required to find what 300*l.* half-yearly pension would amount to in five years at 4 *per cent.* interest: then the simple interest of 1*l.* for half a year being = ,02, and the number of payments = 10, we, in this case, have $A = 300$, $r =$,02, and $n = 10$; and consequently m (*by theorem* 1) $= rA + n \times \frac{n-1}{2} \times rA = 3270l.$ which is the value sought. And the like is to be observed in what follows hereafter.

Of the Present Values of Annuities, or Pensions, computed at Simple Interest.

Let
$A =$ the annuity, pension, or yearly rent.
$r =$ the interest of 1*l.* for one year.
$n =$ the number of years.
$v =$ the present value of the annuity.

Then, because the amount of the annuity, in n years, is found above to be $nA + \frac{1}{2}n \, . \, \overline{n-1} \, . \, rA$, and since 1*l.* present money, is equivalent to $1 + nr$ to be received at the end of the time n, we therefore have $1 + nr : 1 :: nA + \frac{1}{2}n \, . \, \overline{n-1} \, . \, rA$ (the said amount) $: \frac{nA + \frac{1}{2}n \, . \, \overline{n-1} \, . \, rA}{1 + nr}$, its required value, in present money. But it may be observed, that this method, given by authors for determining the values of annuities, *according to simple interest*, is, in reality, a particular sort, or species of *compound interest;* since the allowing of interest upon the annuity, as it becomes due, is nothing less than allowing interest upon interest; the annuity itself being, properly, the simple interest, and the capital, from whence it arises, the principal. It is true, the sum $1 + nr$, expressing the amount of 1*l.* is given, strictly speaking, according to *simple interest:* but the conclusion (as a late author * very justly observes) would be more congruous, and answer better, were the same allowances to be made therein as are made in finding the amount of the annuity; that is, were interest upon interest to be taken once and no more. Agreeable to this assumption, r, the interest of 1*l.* being considered as an annuity, its amount in n years (by writing r for A, in the general *formula* above) will be given $= nr + \frac{1}{2}n \cdot \overline{n-1} \cdot r^2$: to which the principal 1*l.* being added, the aggregate $1 + nr + \frac{1}{2}n \cdot \overline{n-1} \cdot r^2$ will therefore be the whole amount of 1*l.* in the time n; and so we shall have $1 + nr + \frac{1}{2}n \cdot \overline{n-1} \cdot r^2 : 1 :: nA + \frac{1}{2}n \cdot \overline{n-1} \cdot rA : \dfrac{2nA + n \cdot \overline{n-1} \cdot rA}{2 + 2nr + n \cdot \overline{n-1} \cdot r^2} = v$, the true value of the annuity, *according to the said hypothesis.* From which equation others may be derived, by means whereof the different values of A, n, and r, may be successively determined. But, as this method of allowing interest upon interest, once and no more, is arbitrary, and the valuation of annuities, according to simple interest, a matter of more speculation than real use, it being not only customary, but also most equitable to allow compound interest in these cases, I shall not stay to exemplify it, but proceed to

The Resolution of the various Cases of Compound Interest, and of Annuities, as depending thereon.

Let $\begin{cases} R = \text{the amount of 1}l.\text{ in one year, } viz.\text{ principal and interest.} \\ P = \text{any sum put out at interest.} \end{cases}$

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* *Mr. Hardy,* in his Annuities.

Let $n =$ the number of years it is lent for.
$a =$ its amount in that time.
$A =$ any annuity forborne n years.
$m =$ its amount.
$v =$ the present value of the annuity for the same time.

Therefore, since one pound, put out at interest, in the first year is increased to R, it will be, as 1 to R, so is R, the sum forborne the second year, to R^2, the amount of one pound in two years; and therefore as 1 to R, so is R^2, the sum forborne the third year, to R^3, the amount in three years: whence it appears that R^n, or R raised to the power whose exponent is the number of years, will be the amount of one pound in those years. But as 1*l.* is to its amount R^n, so is P to (a) its amount, in the same time; whence we have $P \times R^n = a$. Moreover, because the amount of one pound, in n years, is R^n, its increase in that time will be $R^n - 1$; but its interest for one single year, or the annuity answering to that increase, is $R - 1$; therefore, as $R - 1$ to $R^n - 1$, so is A to m. Hence we get $\frac{A \times \overline{R^n - 1}}{R - 1} = m$. Furthermore, since it appears that one pound, ready money, is equivalent to R^n, to be received at the expiration of n years, we have, as R^n to 1, so is $\frac{A \times \overline{R^n - 1}}{R - 1}$ (the sum in arrear) to v, its worth in ready money; and therefore

$$\frac{A \times \overline{1 - \frac{1}{R^n}}}{R - 1} = v.$$

From which three original equations others may be derived, by help whereof the various questions relating to compound interest, annuities in arrear, and the present values of annuities, may be resolved.

Thus, because PR^n is $= a$, there will come out $P = \frac{a}{R^n}$, and $R = \overline{\frac{a}{P}}\Big|^{\frac{1}{n}}$, &c. or by exhibiting the same equations in logarithms (which is the most easy for practice) we shall have

1°. $\text{Log. } a = \text{log. } P + n \times \text{log. } R.$
2°. $\text{Log. } P = \text{log. } a - n \times \text{log. } R.$
3°. $\text{Log. } R = \frac{\text{log. } a - \text{log. } P}{n}.$
4°. $n = \frac{\text{log. } a - \text{log. } P}{\text{log. } R}.$

Which four theorems, or equations, serve for the four cases in compound interest.

Again, since m is $= \frac{A \times \overline{R^n - 1}}{R - 1}$, we shall have

1°. $\text{Log. } m = \text{log. } A + \text{log. } \overline{R^n - 1} - \text{log. } \overline{R - 1}.$
2°. $\text{Log. } A = \text{log. } m - \text{log. } \overline{R^n - 1} + \text{log. } \overline{R - 1}.$
3°. $n = \frac{\text{log. } \overline{mR - m + A} - \text{log. } A}{\text{log. } R}.$
4°. $R^n - \frac{mR}{A} + \frac{m}{A} - 1 = 0.$

To which the various questions relating to annuities in arrear are referred.

Moreover, seeing $A \times \frac{1 - \frac{1}{R^n}}{R - 1}$ is $= v$, we thence have

1°. $\text{Log. } v = \text{log. } A + \text{log. } \overline{1 - \frac{1}{R^n}} - \text{log. } \overline{R - 1}.$
2°. $\text{Log. } A = \text{log. } v + \text{log. } \overline{R - 1} - \text{log. } \overline{1 - \frac{1}{R^n}}.$
3°. $n = \frac{\text{log. } A - \text{log. } \overline{A + v - vR}}{\text{log. } R}.$
4°. $R^{n+1} - \frac{A}{v} + 1 \times R^n + \frac{A}{v} = 0.$

The use of which theorems, respecting the present values of annuities, as well as of the preceding ones, for compound interest and annuities in arrear, will fully appear from the following examples.

Examp. 1. To find the amount of 575*l.* in seven years, at four *per cent. per annum*, compound interest.

In this case we have given $P = 575$, $R = 1{,}04$, and $n = 7$; therefore, *by theorem* 1, $\log. a = \log. 575 + 7 \log. 1{,}04 = 2{,}8789011$; and consequently $a = 756{,}66$, or 756*l.* 13*s.* $2\frac{1}{4}$*d.* the value required.

Examp. 2. What principal, put to interest, will raise a stock of 1000*l.* in fifteen years, at 5 *per cent.?*

Here we have given $R = 1{,}05$, $n = 15$, and $a = 1000$; therefore, *by theorem* 2, $\log. P = \log. 1000 - 15 \log. 1{,}05 = 2{,}6821605$; and consequently $P = 481{,}02$ or 481*l.* 0*s.* $4\frac{3}{4}$*d.* the value sought.

Examp. 3. In how long time will 575*l.* raise a stock of 756*l.* 13*s.* $2\frac{1}{4}$*d.* at 4 *per cent.?*

In this case we have $R = 1{,}04$, $P = 575$, and $a = 756{,}66$; whence, *by theor.* 4, $n = \frac{\log. 756{,}66 - \log. 575}{\log. 1{,}04} = 7$, the number of years required.

Examp. 4. To find at what rate of interest 481*l.* in fifteen years, will raise a stock of 1000*l.*

Here we have given $P = 481$, $a = 1000$, and $n = 15$; therefore, *by theorem* 3, $\log. R = \frac{\log. 1000 - \log. 481}{15} = .0211903$, whence $R = 1{,}05$; consequently 5 *per cent.* is the rate required.

The four last examples relate to the cases in compound interest; the four next are upon the forbearance of annuities.

Examp. 1. If 50*l.* yearly rent, or annuity, be forborne seven years, what will it amount to, at 4 *per cent. per annum*, compound interest?

Here we have $R = 1{,}04$, $A = 50$, and $n = 7$; and therefore, *by theor.* 1, $\log.\, m\ (= \log.\, A + \log.\, \overline{R^n - 1} - \log.\, \overline{R - 1}) = \log.\, 50 + \log.\, \overline{1{,}04]^7 - 1}, - \log.\, {,}04 = 2{,}596597$; and consequently $m = 395$*l.* the value that was to be found.

Examp. 2. What annuity, forborne seven years, will amount to, or raise a stock of 395*l.* at 4 *per cent.* compound interest?

In this case we have given $R = 1{,}04$, $n = 7$, and $m = 395$; whence, *by theorem* 2,l og. $A\ (= \log.\, m - \log.\, \overline{R^n - 1} + \log.\, \overline{R - 1}) = \log.\, 395 - \log.\, \overline{1{,}04]^7 - 1} + \log.\, {,}04 = 1{,}6989700$; and consequently $A = 50$*l.* which is the annuity required.

Examp. 3. In how long time will 50*l.* annuity raise a stock of 395*l.* at 4 *per cent. per annum*, compound interest?

Here we have $R = 1{,}04$, $A = 50$, $m = 395$; and therefore, *by theor.* 3, $n\ (= \dfrac{\log.\, \overline{mR - m + A} - \log.\, A}{\log.\, R})$

$= \dfrac{{,}1192559}{{,}0170333} = 7$, the number of years required.

Examp. 4. If 120*l.* annuity, forborne eight years, amount to, or raise a stock of 1200*l.* what is the rate of interest?

In this case we have given $n = 8$, $A = 120$, and $m = 1200$, to find R; therefore, *by theorem* 4, we have $R^8 - 10R + 9 = 0$, from which, by any of the methods in sect. 13, the required value of R will be found $= 1{,}06287$; therefore the rate is 6,287, or 6*l.* 5*s.* 9*d.* *per cent. per annum.*

The solution of the last case, where the rate is required, being a little troublesome, I shall here put down an approximation (derived from the third general *formula*, at *p.* 165) which will be found to answer very near the truth, provided the number of years is not very great.

Let $Q = \frac{n\,.\,\overline{n-1}\,.\,A}{2\,.\,\overline{m-nA}}$; then will

$$\frac{3000Q + \overline{2n-1}\,.\,400}{6Q\,.\,\overline{5Q+3n-4} + \frac{1}{6}\,.\,\overline{n-2}\,.\,\overline{11n-13}}$$ be the rate *per cent.* required.

Thus, for example, let $n = 8$, $A = 120$, and $m = 1200$; then will $Q = \frac{56\ .\ 120}{2\ .\ 240} = 14$, and the rate itself $= \frac{42000 + 6000}{84 \times 90 + 75} = 6{,}287$, *as above.*

The preceding examples explain the different cases of *annuities in arrear;* in the following ones the rules for the *valuation of annuities* are illustrated.

Examp. 1. To find the present value of 100*l.* annuity, to continue seven years, allowing 4 *per cent. per annum*, compound interest.

Here we have given $R = 1{,}04$, $A = 100$, and $n = 7$; and therefore, *by theorem* 1, $\log. v\ (= \log. A + \log. \overline{1 - \frac{1}{R^n}} - \log. \overline{R-1}) = \log. 100 + \log. \overline{1 - \frac{1}{\overline{1{,}04}|^7}} - \log. {,}04 = 2{,}778296$; and consequently $v = 600{,}2 = 600$*l.* 4*s.* which is the value that was to be found.

Examp. 2. What annuity, or yearly income, to continue 20 years, may be purchased for 1000*l.* at $3\frac{1}{2}$ *per cent.?*

In this case, $R = 1{,}035$, $n = 20$, $v = 1000$; whence, *by theorem* 2, we have $\log. A\ (= \log. v + \log. \overline{R-1} - \log. \overline{1 - \frac{1}{R^n}}) = 1{,}847336$; and consequently $A = 70{,}36$, or 70*l.* 7*s.* 2*d.*

Examp. 3. For how long time may one, with 600*l.* purchase an annuity of 100*l.* at 4 *per cent.*?

In this example, we have $R = 1{,}04$, $A = 100$, and $v = 600$; and therefore, *by theorem* 3, n $\left(= \frac{\log.\ A - \log.\ \overline{A + v - vR}}{\log.\ R}\right) = 7$, the number of years required.

Examp. 4. To determine at what rate of interest an annuity of 50*l.* to continue ten years, may be purchased, for 400*l.*

Here $A = 50$, $n = 10$, and $v = 400$; whence, *by theorem* 4, $R^{n+1} - \overline{\frac{A}{v} + 1} \times R^n + \frac{A}{v}$ being $= 0$, we have $R^{11} - 1{,}125R^{10} + {,}125 = 0$; which equation resolved, gives the required value of $R = 1{,}042775$; and consequently the rate of interest 4,2775*l.* *per annum.*

The solution of this last case being somewhat tedious, the following approximation (which will be found to answer very near the truth, when the number of years is not very large) may be of use.

Assume $Q = \frac{n \,.\, \overline{n+1} \,.\, A}{2nA - 2v}$; so shall $\frac{3000Q - \overline{2n+1} \times 400}{6Q \,.\, \overline{5Q - 3n - 4} + \frac{1}{6} \,.\, \overline{n+2} \,.\, \overline{11n+13}}$ express the rate *per cent.* very nearly.

Thus, for example, let A (as above) be $= 50$, $n = 10$, and $v = 400$; then, Q being $= \frac{10 \times 11 \times 50}{1000 - 800} = 27.5$, we have $\frac{82500 - 8400}{165 \times 103{,}5 + 246}$, or 4,2775, for the rate *per cent.* the same as before.

SECTION XVII.

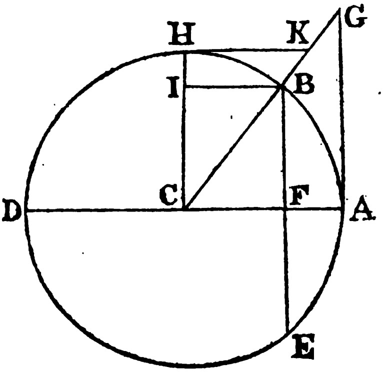
Of Plane Trigonometry.

DEFINITIONS.

1. PLANE trigonometry is the art whereby, having given any three parts of a plane triangle (except the three angles), the rest are determined. In order to which, it is not only requisite that the peripheries of circles, but also that certain right lines, in and about the circle, be supposed divided into some assigned number of equal parts.

2. The periphery of every circle is supposed to be divided into 360 equal parts, called degrees; and each degree into 60 equal parts, called minutes; and each minutes into 60 equal parts, called seconds, or second-minutes, &c. Any part of the periphery is called an arch, and is measured by the number of degrees and minutes, &c. it contains.

3. The difference of any arch from 90 degrees, or a quadrant, is called its complement, and its difference from 180 degrees, or a semicircle, its supplement.



4. A chord, or subtense, is a right line drawn from one extremity of an arch to the other; thus BE is the chord or subtense of the arch BAE or BDE.

5. The sine (or right-sine) of an arch is a right line drawn from one extremity of the arch perpendicular to the diameter passing through the other extremity: thus, BF is the sine of the arch AB, or BD.

6. The versed-sine of an arch is the part of the diameter intercepted between the arch and its sine: so AF is the versed-sine of AB, and DF of DB.

7. The co-sine of an arch is the part of the diameter intercepted between the center and the sine; and is equal to the sine of the complement of that arch. Thus, CF is the co-sine of the arch AB, and is equal to BI, the sine of its complement HB.

8. The tangent of an arch is a right line touching the circle in one extremity of that arch, continued from thence, to meet a line drawn from the center through the other extremity; which line is called the secant of the same arch: thus AG is the tangent, and CG the secant of the arch AB.

9. The co-tangent and co-secant of an arch are the tangent and secant of the complement of that arch: thus HK and CK are the co-tangent and co-secant of the arch AB.

10. A trigonometrical canon is a table exhibiting the lengths of the sine, tangent, &c. to every degree and minute of the quadrant, with respect to the radius, which is supposed unity, and conceived to be divided into 10000000 or more decimal parts. Upon this table the numerical solution of the several cases in trigonometry depends; it will therefore be proper to begin with its construction.

PROPOSITION I.

The number of degrees and minutes, &c. *in an arch being given; to find both its sine and co-sine.*

This problem is resolved, by having the ratio of the circumference to the diameter, and by means of the known series for the sine and co-sine (hereafter demonstrated). For, the semi-circumference of the circle, whose radius is unity, being 3,141592653589793, &c. it will therefore be, as the number of degrees or minutes in the whole semicircle is to the degrees or minutes in the arch proposed, so is 3,14159265358, &c. to the length of the said arch; which let be denoted by a; then, *by the series above quoted,* its sine will be expressed by $a - \frac{a^3}{2.3} + \frac{a^5}{2.3.4.5} - \frac{a^7}{2.3.4.5.6.7}$, *&c.* and its co-sine by $1 - \frac{a^2}{2} + \frac{a^4}{2.3.4} - \frac{a^6}{2.3.4.5.6} + \frac{a^8}{2.3.4.5.6.7.8}$, *&c.*

Thus, for example, let it be required to find the sine of one minute: then, as 10800 (the minutes in 180 degrees) : 1 : : 3,14159265358, *&c.* : .000290888208665 = the length of an arch of one minute: therefore, in this case, $a = .000290888208665$, and $\frac{a^3}{2.3}$ $(= \frac{a^3}{6})$ $= .000000000004102$, *&c.* And, consequently, .000290888204563 = the required sine of one minute.

Again, let it be required to find the sine and co-sine of five degrees, each true to seven places of decimals. Here ,0002908882, the length of an arch of 1 minute (found above), being multiplied by 300, the number of minutes in 5 degrees, the product .08726646 will be the length of an arch of 5 degrees: therefore, in this case we have

$$a = ,08726646,$$
$$-\frac{a^3}{6} = -,00011076,$$
$$+\frac{a^5}{120} = +,00000004,$$

&c. and consequently ,08715574 = the sine of 5 degrees. Also,

$$\frac{a^2}{2} = ,00380771,$$
$$\frac{a^4}{24} = ,00000241;$$

and consequently ,9961947 = the co-sine of 5 degrees.

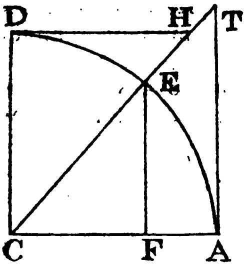
After the same manner, the sine and co-sine of any other arch may be derived; but the greater the arch is, the slower the series will converge, and therefore a greater number of terms must be taken to bring out the conclusion to the same degree of exactness.

But there is another method of constructing the trigonometrical canon; which, though less direct, is more geometrical; and that is by determining the sines and tangents of different arches, one from another, as in the ensuing propositions.

PROPOSITION II.

The sine of an arch being given; to find its co-sine, tangent, co-tangent, secant, and co-secant.

Let AE be the proposed arch, EF its sine, CF its co-sine, AT its tangent, DH its co-tangent, CT its secant, and CH its co-secant: then (*by Euc.* 47. 1.) we shall have $CF = \sqrt{CE^2 - EF^2}$; from whence the co-sine will be known; and then, by reason of the similar triangles, CFE, CAT, and CDH, it will be,



1. CF : FE : : CA : AT; whence the tangent is known.

2. CF : CE : : CA : CT; whence the secant is known.

3. EF : CF : : CD : DH; whence the co-tangent is known.

4. EF : CE : : CD : CH; whence the co-secant is also known.

Hence it appears,

1. That the tangent is a fourth proportional to the co-sine, the sine, and the radius.

2. That the secant is a third proportional to the co-sine and the radius.

3. That the co-tangent is a fourth proportional to the sine, the co-sine, and the radius.

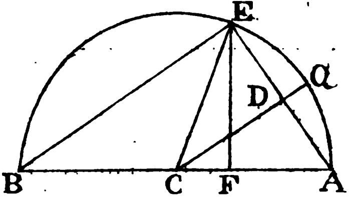
4. That the co-secant is a third proportional to the sine and the radius.

5. And that the rectangle of the tangent and co-tangent is equal to the square of the radius.

PROPOSITION III.

The co-sine CF *of an arch* AE *being given; to find the sine and co-sine of half that arch.*

From the two extremities of the diameter AB draw the subtenses AE and BE; and let CQ bisect the arch AE in Q and its chord (perpendicularly) in D; then, since the angle BEA is a right one (*by Euc.* 31. 3.), the triangles ABE and ADC are similar; and, therefore, AC being = $\frac{1}{2}$ AB, AD must be = $\frac{1}{2}$ AE, and CD = $\frac{1}{2}$ BE: but AE is = $\sqrt{AB \times AF}$, and BE = $\sqrt{AB \times BF}$; therefore



$$\left.\begin{array}{l} AD = \frac{1}{2}\sqrt{AB \times AF} = \sqrt{\frac{1}{2}AC \times AF} = \text{the sine} \\ CD = \frac{1}{2}\sqrt{AB \times BF} = \sqrt{\frac{1}{2}AC \times BF} = \text{the co-sine} \end{array}\right\} \text{of } \frac{1}{2}\ AE.$$

Hence it is evident, that the sine of the half of any arch is a mean proportional between the half radius and the versed-sine of the whole arch; and its co-sine a mean proportional between half the radius and the versed-sine of the supplement of the same arch.

PROPOSITION IV.

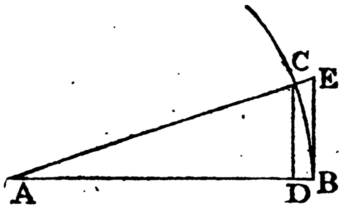
The sine AD, *and co-sine* CD, *of an arch* AQ *being given; to find* EF, *the sine of the double of that arch.* (*See the preceding figure*).

Since AE = 2AD, and BE = 2CD, and the triangles ABE and AEF are alike (*by Euc.* 8. 6.), we have, as AB (2AC) : AE (2AD) : : BE (2CD) : EF; whence it appears, that the sine of double any arch is a fourth proportional to the radius, the sine, and the double-co-sine of the same arch.

PROPOSITION V.

The sine CD, *and tangent* BE, *of a very small arch, are nearly in the ratio of equality.*

For, the triangles ADC and ABE being similar, thence will AD : AB : : DC : BE : but as the point C approaches to B, the difference of AB and AD will become indefinitely small in respect of AB, and therefore the difference of BE and DC will likewise become indefinitely small with respect to BE or DC.



Corollary. Because any arch BC is greater than its sine and less than its tangent; and since the sine and tangent of a very small arch are proved to be nearly equal, it is manifest that a very small arch and its sine are also nearly in the ratio of equality.

PROPOSITION VI.

To find the sine of an arch of one minute.

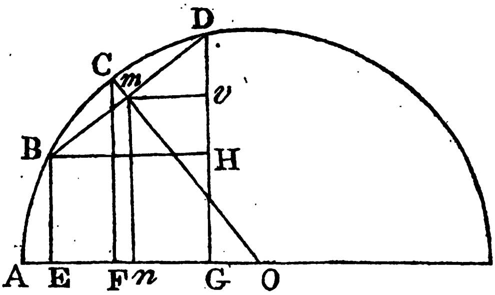
The sine of 30 degrees is known, being half the chord of 60 degrees, or half the radius; therefore, *by prop.* 2. *and* 3. the sine of 15 degrees will be known; and, the sine of 15 degrees being known, the sine of $7^\circ\ 30'$ will be found (*by the same propositions*), and from thence the sine of $3^\circ\ 45'$; and so likewise the sine of half this; and so on, till 12 bisections being made, we come, at last, to the sine of an arch of $52''$, $44'''$, $03''''$, $45'''''$; which sine (*by corol. to the preceding prop.*) will, as the co-sine is nearly equal to the radius, be nearly equal to the arch itself. Therefore we have, as $52''$, $44'''$, $03''''$, $45'''''$, is to $1'$, so is the length of the former of these arches (found as above) to the length of an arch of one minute, or that of its sine, very nearly.

If it be taken for granted, that 3,1415926535, &c. is the length of half the periphery of the circle whose radius is unity, we shall have, as 10800, the number of minutes in 180°, or the whole semicircle, is to one minute, so is 3,1415926535, &c. the whole semicircle, to 0,000290888208, the length of an arch of one minute, or that of its sine, very nearly.

PROPOSITION VII.

If there be three equidifferent arches AB, BC, *and* AD, *it will be, as the radius is to the co-sine of their common difference* BC *or* CD, *so is the sine* CF, *of the mean, to half the sum of the sines* BE + DG, *of the two extremes; and as the radius is to the sine of the common difference, so is the co-sine* FO *of the mean, to half the difference of the sines of the two extremes.*

For, let BD be drawn, cutting the radius OC in m; also draw mn parallel to CF, meeting AO in n, and BH and mv parallel to AO, meeting DG in H and v: then, because the arches BC and CD are equal to each other, OC is not only perpendicular to BD, but also bisects it (*Euç*. 3. 3.); whence it is evident that Bm, or Dm, will be the sine of BC or CD, and Om its co-sine; and that mn, being an arithmetical mean between the sines BE and DG, of the two extremes, is equal to half their sum, and Dv equal to half their difference. Moreover, by reason of the similarity of the triangles OCF, Omn, and Dmv, it will



be as OC : Om : : CF : mn and as OC : Dm : : EO : Dv } Q. E. D.

COROL. 1.

Since, from the foregoing proportions, mn is $= \frac{Om \times CF}{OC}$, and D$v$ (= vH) $= \frac{Dm \times FO}{OC}$, it is evident that DG ($= mn + \mathrm{D}v$) will be $= \frac{\mathrm{O}m \times \mathrm{CF} + \mathrm{D}m \times \mathrm{FO}}{\mathrm{OC}}$, and BE ($= mn - v\mathrm{H}$) $= \frac{\mathrm{O}m \times \mathrm{CF} - \mathrm{D}m \times \mathrm{FO}}{\mathrm{OC}}$: from whence it appears that the sine (DG) of the sum (AD) of any two arches (AC and CD) is equal to the sum of the rectangles of the sine of the one into the co-sine of the other, alternately, divided by the radius; and that the sine (BE) of their difference (AB) is equal to the difference of the same rectangles, divided also by the radius.

COROL. 2.

Moreover, seeing DG + BE ($2mn$) is $= \frac{2\mathrm{O}m \times \mathrm{CF}}{\mathrm{OC}}$, and DG — BE ($= \mathrm{DH} = 2\mathrm{D}v$) $= \frac{2\mathrm{D}m \times \mathrm{FO}}{\mathrm{OC}}$, from the former of these, we have $\mathrm{DG} = \frac{2\mathrm{O}m \times \mathrm{CF}}{\mathrm{OC}} - \mathrm{BE}$, and from the latter, $\mathrm{DG} = \frac{2\mathrm{D}m \times \mathrm{FO}}{\mathrm{OC}} + \mathrm{BE}$; which equations, expressed in words, give the following theorems.

Theor. 1. *If the sine of the mean of three equidifferent arches (supposing the radius unity) be multiplied by twice the co-sine of the common difference, and the sine of either extreme be subtracted from the product, the remainder will be the sine of the other extreme.*

Theor. 2. *Or, if the co-sine of the mean be multiplied by twice the sine of the common difference, and the product be added to or subtracted from the sine of one of the extremes, the sum or remainder will be the sine of the other extreme.*

These two theorems are of excellent use in the construction of the trigonometrical canon: for, supposing the sine and co-sine of an arch of 1 minute to be found, by prop. 6 and 1, and to be denoted by p and q, respectively; then, the sine of 2 minutes being given from prop. 4, the sine of 3 minutes will from hence be known, being $= 2q \times \text{sine } 2' - \text{sine } 1'$ (*by theor.* 1) or $= 2p \times \text{co-sine of } 2' + \text{sine of } 1'$ (*by theor.* 2). After the same manner the sine of 4′ will be found, being $= 2q \times$ sine of 3′ — sine of 2′, or $= 2p \times$ co-sine of 3′ + sine of 2′. And thus the sines of 5, 6, 7, *&c.* minutes may be successively derived by either of the theorems; but the former is the most commodious.

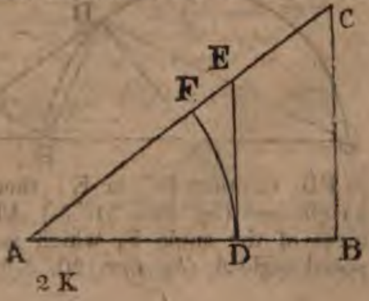
If the mean arch be 45°, then, its co-sine being $= \sqrt{\frac{1}{2}}$, it follows (*from theorem* 2) that the sine of the excess of any arch above 45°, multiplied by $2\sqrt{\frac{1}{2}}$ or $\sqrt{2}$, gives the excess of the sine of this arch above that of another arch as much below 45°; thus, $\sqrt{2} \times$ sine of 10° = sine of 55° — sine of 35°; and $\sqrt{2} \times$ sine of 15° = sine of 60° — sine of 30°; and so of others: which is useful in finding the sines of arches greater than 45°.

But, if the mean arch be 60 degrees, then its co-sine being $\frac{1}{2}$, it is evident, *from the same theorem*, that the sine of the excess of any arch above 60°, added to the sine of another arch as much below 60°, will give the sine of the first arch, or greater extreme: thus, the sine of 10° + sine 50° = sine 70°, and sine 15° + sine 45° = sine 75°; from whence the sines of all arches above 60 degrees, those of the inferior arches being known, are had by addition only.

PROPOSITION VIII.

In any right-angled plane triangle ABC, *it will be, as the base* AB *is to the perpendicular* BC, *so is the radius* (*of the tables*) *to the tangent of the angle at the base.*

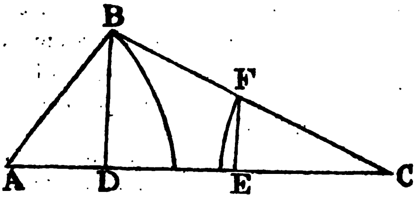
Let DA be the radius to which the table of sines and tangents is adapted, and DE the tangent of the angle A; then, by reason of the similarity of the triangles ABC and ADE, it will be, as AB : BC : : AD : DE. *Q. E. D.*



PROPOSITION IX.

In every plane triangle, it will be, as any one side is to the sine of the opposite angle, so is any other side to the sine of its opposite angle.

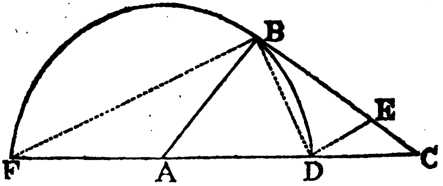
For, let ABC be the proposed triangle; take CF = AB, and upon AC let fall the perpendiculars BD and EF; which will be the sines of the angles A and C, to the equal radii AB and CF. But the triangles CBD and CFE are similar, and therefore CB : BD : : CF (AB) : FE; that is, as CB is to the sine of A, so is AB to the sine of C, *Q. E. D.*



PROPOSITION X.

In every plane triangle, it will be, as the sum of any two sides is to their difference, so is the tangent of the complement of half the angle included by those sides, to the tangent of the difference of either of the other two angles and the said complement.

For, let ABC be the triangle, and AB and AC the two proposed sides; and upon A, as a center, with the radius AB, let a semicircle be described, cutting CA produced, in D and F; so that CF may express the sum, and CD the difference of the sides AC and AB; join F, B, and B, D, and draw DE parallel to FB, meeting BC in E; then the angle FBD being a right one (*by Euc.* 31. 3.) ADB will be the complement of the angle F, which is equal to half the proposed angle A (*by Euc.* 20. 3.). Moreover, seeing the

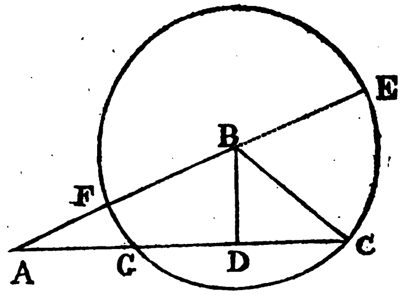


angles FBD and EDB are both right ones, for EDB is = FBD (= a right angle), because DE is parallel to FB, it is plain, that, if BD be made the radius, BF will be the tangent of BDF, and DE the tangent of DBE: but, because of the similar triangles CFB and CDE, CF : CD : : BF : DE; that is, as the sum of the sides AC and AB is to their difference, so is the tangent of BDF to the tangent of DBC; which angle is manifestly the excess of ABC above BDF, or ABD; and also the excess of ADB above ACB. *Q. E. D.*

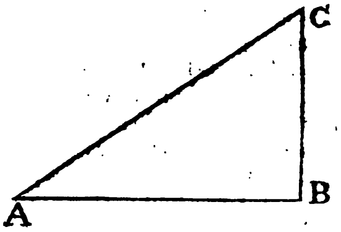
PROPOSITION XI.

As the base of any plane triangle is to the sum of the two sides, so is the difference of the sides to the difference of the segments of the base, made by a perpendicular falling from the vertical angle.

For, let ABC be the proposed triangle, and BD the perpendicular; from B as a center, with the interval BC, let the circumference of a circle be described, cutting the base AC in G, and the side AB, produced, in F and E: then will AE be the sum of the sides, AF their difference, and AG the difference of the segments of the base AD and DC: but (*by Euc.* 36. 3.) AE × AF = AC × AG; and therefore AC : AE : : AF : AG. *Q. E. D.*

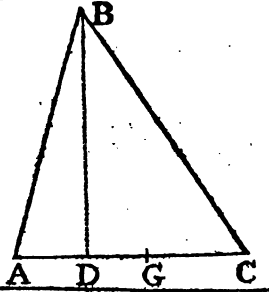


The solution of the cases of right-angled plane triangles.



Case	Given.	Sought.	Proportion.
1	The hypothenuse AC and the angles.	The leg BC.	As the radius (or the sine of B) is to the hyp. AC; so is the sine of A, to its opposite side BC (*by prop.* 9.)
2	The hypoth. AC and one leg AB.	The angles.	As AC : rad. : : AB : sine of C; whose complement gives the angle A.
3	The hypoth. AC and one leg AB.	The other leg BC.	Let the angles be found by case 2; then, as rad. : AC : : sine of A : BC (*by prop.* 9.)
4	The angles and one leg AB.	The hypothenuse AC.	As sine of C : AB : : rad. (sine of B) : AC (*by prop.* 9.)
5	The angles and one leg AB.	The other leg BC.	As sine of C : AB : : sine of A : BC (*by prop.* 9.) Or, rad. : tang. of A : : AB : BC (*by prop* 8.)
6	The two legs AB and BC.	The angles.	As AB : BC : : rad. : tang. of A (*by prop.* 8); whose complement gives the angle C.
7	The two legs AB and BC.	The hypothenuse AC.	Find the angles by case 6, and from thence the hyp. AC, by case 4.

The solution of the cases of oblique plane triangles.



Case	Given.	Sought.	Proportion.
1	The angles and one side AB.	Either of the other sides, suppose BC.	As sine of C : AB : : sine of A : BC (*by prop.* 9.)
2	Two sides AB, BC and the angle C opposite to one of them.	The other angles A and ABC	As AB : sine of C : : BC : sine of A; which added to C, and the sum subtracted from 180°, gives the angle ABC.
3	Two sides AB, BC and the angle C opposite to one of them.	The other side AC.	Find the angle ABC by case 2; then, as sine of A : BC : : sine of ABC : AC.
4	Two sides AB, AC and the included angle A.	The other angle C and ABC.	As AB+AC : AB—AC : : tang. of the comp. of ½ A : tang. of an ang. which added to the said com. gives the greater ang. C; and subtracted leaves the lesser ABC (*prop.* 10.)
5	Two sides AB, AC and the included angle A,	The other side BC.	Find the angles by case 4; and then BC, by case 1.
6	All the sides.	An angle, suppose A	Let fall a perpendicular BD, opposite the required angle, and suppose DG = AD; then (*by prop.* 11.) AC : BC + BA : : BC — BA : CG, which subtracted from AC, and the remainder divided by 2, gives AD; whence A will be found, by case 2, of right angles.

SECTION XVIII.

The Application of Algebra to the Solution of Geometrical Problems.

WHEN a geometrical problem is proposed to be resolved by algebra, you are, in the first place, to describe a figure that shall represent or exhibit the several parts or conditions thereof, and look upon that figure as the true one; then, having considered attentively the nature of the problem, you are next to prepare the figure for a solution (if need be), by producing and drawing such lines therein as appear most conducive to that end. This done, let the unknown line, or lines, which you think will be the easiest found (whether required or not), together with the known ones (or as many of them as are requisite), be denoted by proper symbols; then proceed to the operation, by observing the relation that the several parts of the figure have to each other; in order to which, a competent knowledge in the elements of geometry is absolutely necessary.

As no general rule can be given for the drawing of lines, and electing the most proper quantities to substitute for, so as always to bring out the most simple conclusions (because different problems require different methods of solution), the best way, therefore, to gain experience in this matter is to attempt the solution of the same problem several ways, and then apply that which succeeds best to other cases of the same kind, when they afterwards occur. I shall, however, subjoin a few general directions, which will be found of use.

1°. In preparing the figure, by drawing lines, let them be either parallel or perpendicular to other lines in the figure, or so as to form similar triangles; and if an angle be given, let the perpendicular be opposite to that angle, and also fall from the end of a given line, if possible.

2°. In electing proper quantities to substitute for, let those be chosen (whether required or not) which lie nearest the known or given parts of the figure, and by help whereof the next adjacent parts may be expressed, without the intervention of surds, by addition and subtraction only. Thus, if the problem were to find the perpendicular of a plane triangle, from the three sides given, it will be much better to substitute for one of the segments of the base, than for the perpendicular, though the quantity required; because the whole base being given, the other segment will be given, or expressed, by subtraction only, and so the final equation come out a simple one; from whence the segments being known, the perpendicular is easily found by common arithmetic: whereas, if the perpendicular were to be first sought, both the segments would be surd quantities, and the final equation an ugly quadratic one.

3°. When, in any problem, there are two lines or quantities alike related to other parts of the figure, or problem, the best way is to make use of neither of them, but to substitute, for their sum, their rectangle, or the sum of their alternate quotients, or for some line or lines in the figure, to which they have both the same relation. This rule is exemplified in prob. 22, 23, 24, and 27.

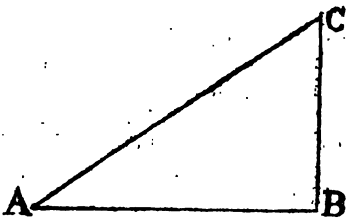
4°. If the area, or the perimeter of a figure be given, or such parts thereof as have but a remote relation to the parts required, it will sometimes be of use to assume another figure similar to the proposed one, whereof one side is unity, or some other known quantity; from whence the other parts of this figure, by the known proportions of the homologous sides, or parts, may be found, and an equation obtained, as is exemplified in prob. 25 and 32.

These are the most general observations I have been able to collect; which I shall now proceed to illustrate by proper examples.

PROBLEM I.

The base (b), and the sum of the hypothenuse and perpendicular (a), of a right-angled triangle, ABC, being given; to find the perpendicular.

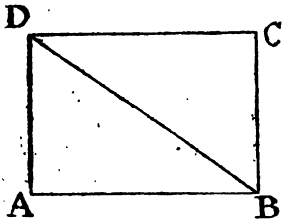
Let the perpendicular BC be denoted by x; then the hypothenuse AC will be expressed by $a-x$: but (*by Euc.* 47. 1.) $AB^2+BC^2=AC^2$; that is, $b^2+x^2=a^2-2ax+xx$; whence $x=\frac{a^2-b^2}{2a}$ = the perpendicular required.



PROBLEM II.

The diagonal and the perimeter of a rectangle, ABCD, *being given; to find the sides.*

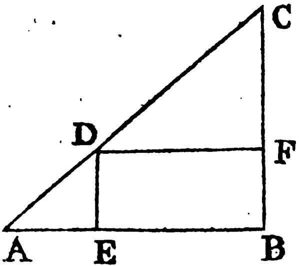
Put the diagonal BD $=a$, half the perimeter (DA + AB) $=b$, and AB $=x$; then will AD $=b-x$; and, therefore, AB^2+AD^2 being $=BD^2$, we have $x^2+b^2-2bx+x^2=a^2$; which, solved, gives $x=\frac{\sqrt{2a^2-b^2}+b}{2}$.



PROBLEM III.

The area of a right-angled triangle ABC, *and the sides of a rectangle,* EBDF, *inscribed therein, being given; to determine the sides of the triangle.*

Put DF $=a$, DE $=b$, BC $=x$, and the measure of the given area ABC $=d$: then, by similar triangles, we shall have $x-b$ (CF) : a (DF) : : x (BC) : AB $=\frac{ax}{x-b}$. Therefore $\frac{ax}{x-b}\times\frac{x}{2}=d$, and consequently $ax^2=2dx-2bd$, or $x^2-\frac{2dx}{a}=-\frac{2bd}{a}$: which, solved,

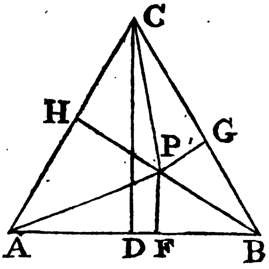


gives $x = \frac{d}{a} \pm \sqrt{\frac{dd}{aa} - \frac{2bd}{a}}$, from whence AB and AC will likewise be known.

PROBLEM IV.

Having the lengths of the three perpendiculars PF, PG, PH, *drawn from a certain point* P *within an equilateral triangle* ABC, *to the three sides thereof; from thence to determine the sides.*

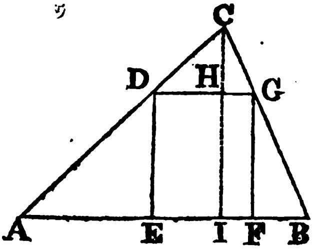
Let lines be drawn from P to the three angles of the triangle; and let CD be perpendicular to AB: call PF a; PG b; PH c; and AD $= x$: then will AC $(= \text{AB}) = 2x$, and CD $(= \sqrt{\text{AC}^2 - \text{AD}^2}) = \sqrt{3xx} = x\sqrt{3}$; and consequently the area of the whole triangle ABC $(= \text{CD} \times \text{AD}) = xx\sqrt{3}$. But this triangle is composed of the three triangles APB, BPC, and APC; whereof the respective areas are ax, bx, and cx. Therefore we have $xx\sqrt{3} = ax + bx + cx$; and from thence, by division, $x = \frac{a + b + c}{\sqrt{3}}$.



PROBLEM V.

Having the area of a rectangle DEFG *inscribed in a given triangle* ABC; *to determine the sides of the rectangle.*

Let CI be perpendicular to AB, cutting DG in H; and let CI $= a$, AB $= b$, DG $= x$, and the given area $= cc$: then it will be, as $b : x :: a : \frac{ax}{b} =$ CH; which, taken from CI,

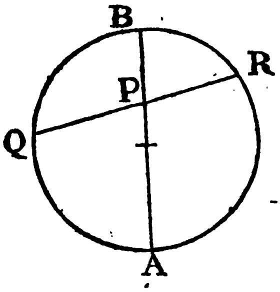


leaves $a - \frac{ax}{b} = IH$; and this, multiplied by x, gives $ax - \frac{ax^2}{b} = cc$ = the area of the rectangle; whence we have $abx - ax^2 = bcc$, $x^2 - bx = -\frac{bcc}{a}$, $x - \frac{b}{2} = \pm \sqrt{\frac{b^2}{4} - \frac{bcc}{a}}$, and $x = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} - \frac{bcc}{a}}$.

PROBLEM VI.

Through a given point P, *within a given circle, so to draw a right line, that the two parts thereof,* PR, PQ, *intercepted between that point and the circumference of the circle, may have a given difference*

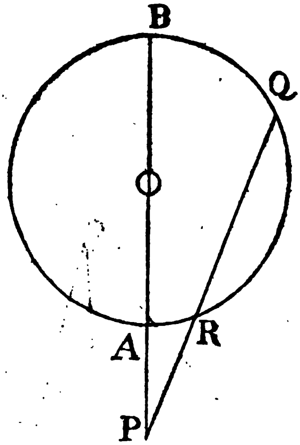
Let the diameter APB be drawn; and let AP and BP, the two parts thereof (which are supposed given) be denoted by a and b; making PR = x, and PQ = $x + d$ (d being the given difference). Then, by the nature of the circle, PQ × PR being = PA × PB, we have $x + d \times x = ab$, or $xx + dx = ab$; whence x is found $= \sqrt{ab + \frac{1}{4}dd} - \frac{1}{2}d$.



PROBLEM VII.

From a given point P, *without a given circle, so to draw a right line* PQ, *that the part thereof,* RQ, *intercepted by the circle, shall be to the external part* PR, *in a given ratio.*

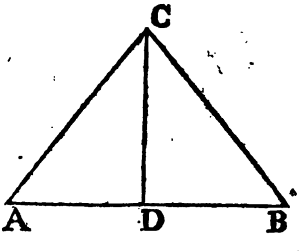
Through the center O, draw PAB; put PA $= a$, PB $= b$, PR $= x$, and let the given ratio of PR to RQ be that of m to n; then it will be, as $m : n :: x : \frac{nx}{m} =$ RQ; therefore PQ $= x + \frac{nx}{m}$; but PR × PQ = PA × PB, or $x \times \overline{x + \frac{nx}{m}} = ab$; therefore $mx^2 + nx^2 = mab$, and $x = \sqrt{\frac{mab}{m+n}}$.



PROBLEM VIII.

The sum of the two sides of an isosceles triangle ABC *being equal to the sum of the base and perpendicular, and the area of the triangle being given; to determine the sides.*

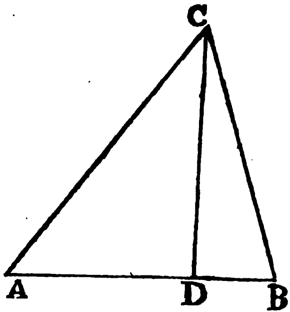
Put the semi-base AD $= x$, the perpendicular CD $= y$, and the given area ABC $= a^2$: so shall $xy = a^2$, and $2\sqrt{xx + yy} = 2x + y$ (*by El.* 47, 1, *and the conditions of the problem*). Now, squaring both sides of the last equation, we have $4xx + 4yy = 4xx + 4xy + yy$; whence $3yy = 4xy$, and consequently $y = \frac{4x}{3}$: which value, substituted in the former equation, gives $\frac{4xx}{3} = a^2$; from whence $x = \sqrt{\frac{3a^2}{4}} = \frac{1}{2}a\sqrt{3}$; $y\ (= \frac{4x}{3}) = \frac{2}{3}a\sqrt{3}$; and AC



$(= \sqrt{xx + yy} = \sqrt{\frac{3aa}{4} + \frac{4aa}{3}} = \sqrt{\frac{25aa}{12}}) =$ $\frac{3}{5}a\sqrt{\frac{1}{3}} = \frac{5}{6}a\sqrt{3}$.

PROBLEM IX.

The segments of the base AD *and* BD, *and the ratio of the sides* AC *and* BC, *of any plane triangle* ABC *being given; to find the sides.*



Put AD $= a$, BD $= b$, AC $= x$; and let the given ratio of AC to BC be as m to n, so shall BC $= \frac{nx}{m}$. But AC^2 — AD^2 (= DC^2) = BC^2 — BD^2, that is, in species, $x^2 - a^2 = \frac{nnxx}{mm} - b^2$. Hence we have $m^2x^2 - n^2x^2 = m^2 \times \overline{aa - bb}$, and $x = m\sqrt{\frac{aa - bb}{mm - nn}}$.

PROBLEM X.

The base AB (a), *the perpendicular* CD $= b$, *and the difference* (d) *of the sides* AC — BC, *of any plane triangle* ABC, *being given; to determine the triangle.* (See the preceding figure.)

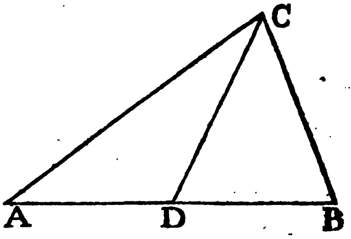
Let the sum of the sides AC + BC be denoted by x: then (*by prop.* 11, *sect.* 18) we shall have $a : x :: d : \frac{dx}{a} =$ the difference of the segments of the base; therefore the greater segment AD will be $= \frac{a}{2} + \frac{dx}{2a} = \frac{aa + dx}{2a}$. But AD^2 + DC^2 = AC^2; that is, $\frac{a^4 + 2a^2dx + d^2x^2}{4aa} + b^2 = \frac{x^2 + 2dx + dd}{4}$: whence

$a^4 + 2a^2dx + d^2x^2 + 4a^2b^2 = a^2x^2 + 2a^2dx + a^2d^2$; which, solved, gives $x = a\sqrt{\frac{aa + 4bb - dd}{aa - dd}}$.

PROBLEM XI.

The base AB, *the sum of the sides* AC + BC, *and the length of the line* CD *drawn from the vertex to the middle of the base, being given; to determine the triangle.*

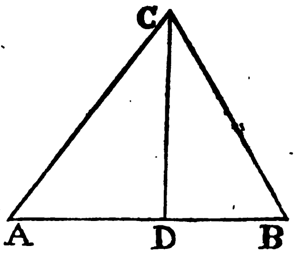
Make AD (= BD) = a, DC = b, AC + BC = c, and AC = x: so shall BC = $c - x$. But AC² + BC² is = 2AD² + 2DC² (*by El.* 12, 2); that is, $x^2 + \overline{c - x}|^2 = 2a^2 + 2b^2$; which, by reduction, becomes $x^2 - cx = a^2 + b^2 - \frac{1}{2}c^2$; whence x is found $= \frac{1}{2}c \pm \sqrt{aa + bb - \frac{1}{4}cc}$.



PROBLEM XII.

The two sides AC, BC, *and the line* CD *bisecting the vertical angle of a plane triangle* ABC, *being given; to find the base* AB.

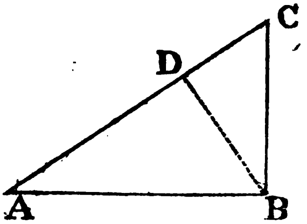
Call AC a; BC b; CD c; and AB x: then $a + b : x :: a :$ AD $= \frac{ax}{a + b}$; and $a + b : x :: b :$ DB $= \frac{bx}{a + b}$. But (*by El.* 20, 3) AC × CB − AD × DB = CD², that is, $ab - \frac{abx^2}{\overline{a \times b}|^2} = c^2$; from whence x will be found $= \overline{a + b} \cdot \sqrt{\frac{ab - cc}{ab}}$.



PROBLEM XIII.

The perimeter AB + BC + CA, *and the perpendicular* BD, *falling from the right angle* B, *to the hypothenuse* AC, *being given; to determine the triangle.*

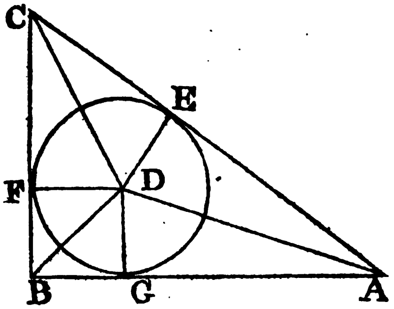
Let BD = a, AB = x, BC = y, AC = z, and AB + BC + CA = b: then, by reason of the similar triangles ACB and ABD, it will be, as $z : y :: x : a$; and therefore $xy = az$: moreover, $x^2 + y^2 = z^2$ (*by Euc.* 47, 1), and $x + y + z = b$ (*by the question*). Transpose z in the last equation, and square both sides, and you will have $x^2 + 2xy + y^2 = b^2 - 2bz + z^2$; from which take $x^2 + y^2 = z^2$, and there will remain $2xy = b^2 - 2bz$; but, by the first equation, $2xy$ is $= 2az$; therefore $2az = b^2 - 2bz$; and $z = \frac{b^2}{2a + 2b}$; whence z is known. But to find x and y from hence, put $\frac{bb}{2a + 2b} = c$, and let this value of z be substituted in the two foregoing equations, $x + y = b - z$, and $zy = az$, and they will become $x + y = b - c$, and $xy = ac$: from the square of the former of which subtract the quadruple of the latter, so shall $x^2 - 2xy + y^2 = \overline{b - c}|^2 - 4ac$; and consequently $x - y = \sqrt{\overline{b - c}|^2 - 4ac}$. This equation being added to, and subtracted from $x + y = b - c$, gives $2x = b - c + \sqrt{\overline{b - c}|^2 - 4ac}$, and $2y = b - c - \sqrt{\overline{b - c}|^2 - 4ac}$.



PROBLEM XIV.

Having the perimeter of a right-angled triangle ABC, *and the radius* DF *of its inscribed circle; to determine all the sides of the triangle.*

From the center D, to the angular points A, B, C, and the points of contact E, F, G, let lines DA, DB, DC, DE, DF, DG be drawn; making DE,



DF, or DG $= a$, AB $= x$, BC $= y$, AC $= z$, and $x + y + z = b$. It is evident that $\frac{ax}{2} + \frac{ay}{2} + \frac{az}{2}$, or its equal $\frac{ab}{2}$ (expressing the sum of the areas ADB, BDC, and ADC) will be $= \frac{xy}{2} =$ the area of the whole triangle ABC; and consequently $2xy = 2ab$: moreover (*by Euc.* 47, 1), $x^2 + y^2 = z^2$; to which if $2xy = 2ab$ be added, we shall have $x^2 + 2xy + y^2$, or $\overline{x + y}|^2 = z^2 + 2ab$; but, by the first step, $\overline{x + y}|^2$ is $= \overline{b - z}|^2 = b^2 - 2bz + z^2$; therefore, by making these two values of $\overline{x + y}|^2$ equal to each other, we get $z^2 + 2ab = b^2 - 2bz + z^2$; whence $2a = b - 2z$, and $z = \frac{1}{2}b - a$. But, to find x and y, from hence, we have now given $x + y$ $(= b - z) = \frac{1}{2}b + a$, and $xy = ab$: the former of these equations, multiplied by x, gives $x^2 + xy = \frac{bx}{2} + ax$; from which the latter $xy = ab$ being subtracted, we have $x^2 = \frac{1}{2}bx + ax - ab$, or $x^2 - \frac{2a + b}{2} \times x = - ab$: whence, by completing the square, &c. $x = \frac{2a + b \pm \sqrt{4a^2 - 12ab + b^2}}{4}$: so that the three sides of the triangle are, $\frac{1}{2}b - a$, $\frac{2a + b + \sqrt{4a^2 - 12ab + b^2}}{4}$, and $\frac{2a + b - \sqrt{4a^2 - 12ab + b^2}}{4}$

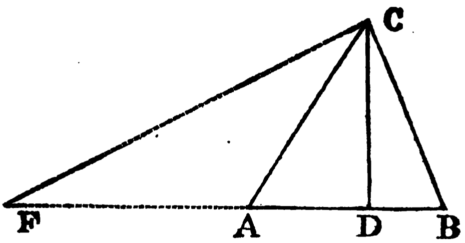
Otherwise.

The right-angled triangles ADE, ADG, having the sides DE, DG equal and AD common, have also AE equal to AG: and, for the like reason, is CE = CF; and consequently AC (AE + CE) = AG + CF. Whence it appears that the hypothenuse is less than the sum of the two legs AB + BC, by the diameter of the inscribed circle, and therefore less than half the perimeter by the semi-diameter of the same circle. Hence we have AC $= \frac{1}{2}b - a$, and AB + BC $= \frac{1}{2}b + a$. Put, therefore, $\frac{1}{2}b - a = c$, $\frac{1}{2}b + a = d$, and half the difference of AB and BC $= x$; then will AB $= d + x$, and BC $= d - x$; and consequently $2d^2 + 2x^2$ ($AB^2 + BC^2$) $= c^2$ (AC^2), whence x is found $= \sqrt{\frac{1}{2}c^2 - d^2}$; therefore AB is $= \frac{1}{2}d + \sqrt{\frac{1}{2}c^2 - d^2}$, and BC $= \frac{1}{2}d - \sqrt{\frac{1}{2}c^2 - d^2}$.

PROBLEM XV.

All the three sides of a triangle ABC *being given; to find the perpendicular, the segments of the base, the area, and the angles.*

Put AC $= a$, AB $= b$, BC $= c$, and the segment AD $= x$; then BD being $= b - x$, we have $c^2 - \overline{b - x}|^2$ ($= CD^2$) $= a^2 - x^2$, that is, $c^2 - b^2 + 2bx - x^2 = a^2 - x^2$; whence $2bx = aa + bb - cc$, and



$x = \frac{aa+bb-cc}{2b}$: now, $CD^2 = AC^2 - AD^2 = \overline{AC+AD} \times \overline{AC-AD} = \overline{a + \frac{aa+bb-cc}{2b}} \times \overline{a - \frac{aa+bb-cc}{2b}} = \frac{2ab+aa+bb-cc}{2b} \times \frac{2ab-aa-bb+cc}{2b} = \frac{\overline{a+b}\rceil^2 - c^2}{2b} \times \frac{c^2 - \overline{a-b}\rceil^2}{2b}$; hence $CD = \frac{1}{2b} \times \sqrt{\overline{a+b}\rceil^2 - c^2 \times \overline{c^2 - \overline{a-b}\rceil^2}}$; and the area $(\frac{CD \times AB}{2}) =$

$$\tfrac{1}{4}\sqrt{\overline{\overline{a+b}\rceil^2 - c^2} \times \overline{c^2 - \overline{a-b}\rceil^2}}.$$

But, because the difference of the squares of any two lines, or numbers, is equal to a rectangle under their sum and difference, the factor $\overline{a+b}\rceil^2 - c^2$ will be $= \overline{a+b+c} \times \overline{a+b-c}$; and the remaining factor $\overline{c^2 - \overline{a-b}\rceil^2} = \overline{c+a-b} \times \overline{c-a+b}$: and so the area will be likewise truly expressed by

$$\tfrac{1}{4}\sqrt{\overline{a+b+c} \times \overline{a+b-c} \times \overline{c+a-b} \times \overline{c-a+b}}$$
$$= \sqrt{\frac{a+b+c}{2} \times \frac{a+b-c}{2} \times \frac{c+a-b}{2} \times \frac{c-a+b}{2}}$$
$$= \sqrt{s \,.\, \overline{s-c} \,.\, \overline{s-b} \,.\, \overline{s-a}}; \text{ by making } s = \frac{a+b+c}{2}.$$

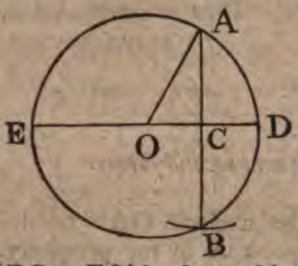
In order to determine the angles, which yet remain to be considered, we may proceed according to prop. 11, in trigonometry, by first finding the segments of the base: but there is another proportion frequently used in practice, which is thus derived: let BA be produced to F, so that AF may be = AC; and then, FC being joined, it is plain that the angle F will be the half of the angle A; and DF (= AC + AD) will be given

For, since (by construction) DE is $= a$, it is plain, if CD be called x, that CE will be $x + a$; but if CE be called x, then CD will be $x - a$: but (*by Euc.* 37. 3.) CE × CD = AC × BC, that is, $\overline{x + a} \times x$ $(x^2 + ax)$ is $= bc$, in the first case; and $x \times \overline{x - a}$ $(x^2 - ax) = bc$, in the second: which two are the very equations above exhibited.

When b and c are equal, the construction will be rather more simple; for, AB vanishing, AC will then coincide with the tangent CF; therefore, if a right-angled triangle OFC be constituted, whose two legs, OF and FC, are equal, respectively, to the given quantities $\frac{1}{2}a$ and b, then will CD (= CD — OF) be the true value of x in the former case, and CE (= CD + OF) its true value in the latter.

Construction of the third form.

With a radius equal to $\frac{1}{2}a$, let a circle be described (as in the two preceding forms), in which apply AB, equal to the sum of the two given quantities $b + c$, and take therein AC equal to either of them; through C draw the diameter DCE; then either DC, or EC, will be the root of the equation.

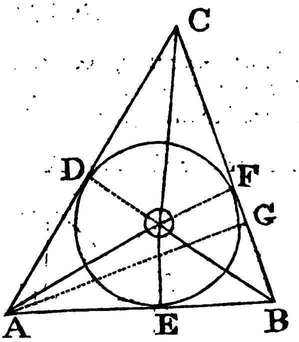


For, the whole diameter ED being $= a$, it is evident that, if either part thereof (DC or EC) be denoted by x, the remaining part will be $a - x$: but DC × EC = AC × CB (*Euc.* 35. 3.), that is, $ax - x^2 = bc$, *as was to be shown.*

The method of construction, when b and c are equal, is no ways different; except that it will be unnecessary to describe the whole circle; for, AC being, here, perpendicular to the diameter ED, if a right-angled triangle OCA be formed, whose hypothenuse is $\frac{1}{2}a$, and one of its legs (AC) $= b$, it is evident that the sum (EC) and the difference (DC) of the hypothenuse and the other leg will be the two values of x required.

PROBLEM XVIII.

The area, the perimeter, and one of the angles of any plane triangle ABC *being given, to determine the triangle.*

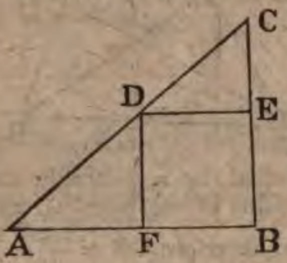


Suppose a circle to be inscribed in the triangle, touching the sides thereof in the points D, E, and F; also from the center O, suppose OA, OD, OC, OF, OB, and OE to be drawn: and upon BC let fall the perpendicular AG; putting AB + BC + AC = b, the given area $= a^2$, the sine of the angle ACB (the radius being 1) $= m$, the co-tangent of half that angle (or the tangent of DOC) $= n$, and AC $= x$. Therefore, since the area of the triangle is equal to $\frac{1}{2}$AB × OE + $\frac{1}{2}$BC × OF + $\frac{1}{2}$AC × OD, that is, equal to a rectangle under half the perimeter and the radius of the inscribed circle, we have $\frac{b}{2} \times \text{OE} = aa$; and therefore $\text{OE} = \frac{2aa}{b}$. But AD being = AE, and BF = BE; it is manifest that the sum of the sides, CA + CB, exceeds the base AB, by the sum of the two equal segments CD and CF; and so is greater than half the perimeter by one of those equal segments CD; that is, CA + CB = $\frac{1}{2}b$ + CD: but (*by trigonometry*) as 1 (radius) : n (the tangent of DOC) : : $\frac{2aa}{b}$ (OD) : DC $= \frac{2na^2}{b}$; whence CA + CB $(= \frac{1}{2}b + \text{CD}) = \frac{1}{2}b + \frac{2na^2}{b}$; which, taken from ($b$) the whole perimeter, leaves $\frac{1}{2}b - \frac{2na^2}{b}$ = the base AB. Make, now, $\frac{1}{2}b + \frac{2na^2}{b} = c$; then will BC $= c - x$; also, (*by trigonometry*) it will be, as 1 (radius) : m (the sine of ACG) : : x (AC) : mx = AG; half whereof, multiplied by $c - x$ (BC), gives $\frac{mcx - mx^2}{2} = a^2$, the area of the triangle: from whence x comes out $= \frac{1}{2}c \pm \sqrt{\frac{c^2}{4} - \frac{2aa}{m}}$.

PROBLEM XIX.

The hypothenuse, AC, *of a right-angled triangle* ABC, *and the side of the inscribed square* BEDF, *being given; to determine the other two sides of the triangle.*

Let DE, or DF $= a$, AC $= b$, AB $= x$, and BC $= y$; then it will be, as $x : y :: x - a$ (AF) : a (FD); whence we have $ax = yx - ya$, and consequently $xy = ax + ay$. Moreover, $xx + yy = bb$: to which equation let the double of the former be added, and there arises $x^2 + 2xy + y^2 = b^2 + 2ax + 2ay$; that is, $\overline{x+y}\rceil^2 = b^2 + 2a \times \overline{x+y}$, or $\overline{x+y}\rceil^2 - 2a \times \overline{x+y} = b^2$; where, by considering $x + y$ as one quantity, and completing the square, we have $\overline{x+y}\rceil^2 - 2a \times \overline{x+y} + a^2 = b^2 + a^2$; whence $\overline{x+y} - a = \sqrt{b^2 + a^2}$, and $x + y = \sqrt{a^2 + b^2} + a$; which put $= c$: then, by substituting $c - x$ instead of its equal (y) in the equation $xy = ax + ay$, there will arise $cx - x^2 = ac$; whence x will be found $= \frac{1}{2}c + \sqrt{\frac{1}{4}cc - ac}$, and $y = \frac{1}{2}c - \sqrt{\frac{1}{4}cc - ac}$.

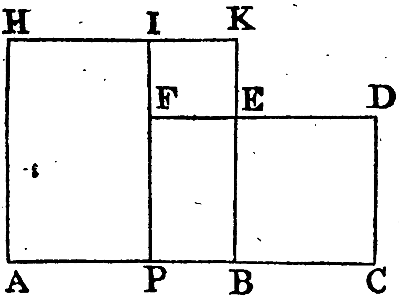


It appears from hence that c, or its equal $\sqrt{aa + bb} + a$, cannot be less than $4a$, and therefore b^2 not less than $8a^2$; because the quantity $\frac{1}{4}cc - ac$, under the radical sign, would be negative, and its square root impossible; it being known that all squares, whether from positive or negative roots, are positive; so that there cannot arise any such things as negative squares, unless the conditions of the problem under consideration be inconsistent and impossible. And this may be demonstrated, from geometrical principles, by means of the following

LEMMA.

The sum of the squares of any two quantities is greater than a double rectangle under those quantities, by the square of the difference of the same quantities.

For, let the greater of the two quantities be represented by AB, and the lesser by BC (both taken in the same right line). Upon AB and BC let the squares AK and CE be constituted; take AP = BC, and complete the rectangles PH and CF. Therefore, because AB = AH, and AP = BC, it is plain that PH and PD are equal to two rectangles under the proposed quantities AB and BC; but these two rectangles are less than the two squares AK and CE, which make up the whole figure, by the square FK, that is, by the square of PB, the difference of the two quantities given; *as was to be proved.*

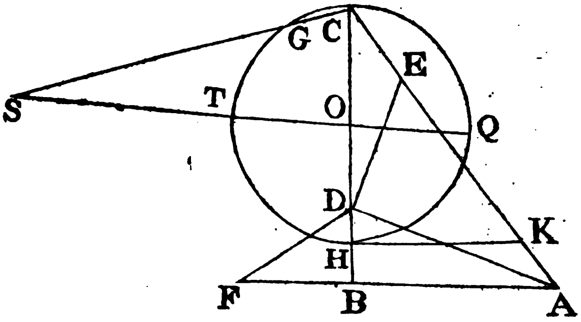


Now, to apply this to the matter proposed, let there be given the quadratic equation $x^2 + b^2 = 2ax$, or $x = a \pm \sqrt{aa - bb}$: then, I say, this equation (and consequently any problem wherein it arises) will be impossible, when $aa - bb$ is negative, or b greater than a. For, since b is supposed greater than a, $2bx$ will likewise be greater than $2ax$; but $2ax$ is given $= xx + bb$, therefore $2bx$ will be greater than $xx + bb$, that is, the double rectangle of two quantities will be greater than the sum of their squares, *which is proved to be impossible.*

PROBLEM XX.

The base AB (a) *and the perpendicular* BC (b) *of a right-angled triangle* ABC, *being given; it is proposed to find a point* D *in the perpendicular, such that, if two right lines be drawn from thence, one to the angular point* A, *and the other* (DE) *perpendicular thereto, the triangles* DEC, ABD, *cut off by those lines, shall be to one another in a given ratio.*

Let AB be produced to F, so that the angle BFD may be equal to the angle BCA; putting AC $= c$, CD $= x$,



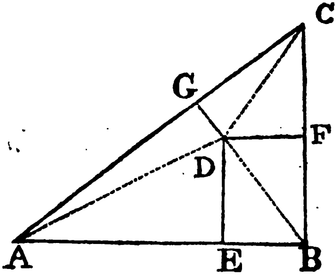
and the given ratio of the triangle DEC to ABD as m to n. Then, by reason of the similar triangles ABC, DBF, it will be, a (AB) : b (BC) : : $b - x$ (BD) : BF $= \frac{b^2 - bx}{a}$; whence AF $= a + \frac{b^2 - bx}{a} = \frac{a^2 + b^2 - bx}{a} = \frac{c^2 - bx}{a}$ (because $a^2 + b^2 = c^2$). Also, as ADE is a right angle, the angles FAD, EDC will be equal; therefore, the angles C and F being equal (*by con.*), the triangles AFD, DCE must be similar; and consequently AF^2 $\left(\frac{\overline{c^2 - bx}|^2}{a^2}\right)$: CD^2 (x^2) : : $\frac{\text{AF} \times \text{BD}}{2}$ $\left(\frac{\overline{b - x} \times \overline{c^2 - bx}}{2a}\right)$, the area of the triangle AFD : $\left(\frac{\overline{b - x} \times ax^2}{2 \times \overline{c^2 - bx}}\right)$, the area of the triangle DEC: wherefore, the area of the triangle ABD being $\frac{BD \times AB}{2}$, or $\frac{\overline{b-x} \times a}{2}$, we shall have $m : n :: \frac{\overline{\overline{b-x} \times a}\,^2}{2 \times \overline{c^2 - bx}} : \frac{\overline{b-x} \times a}{2}$ (*by the question*); and consequently $nx^2 = m \times \overline{c^2 - bx}$, or $x^2 + \frac{mbx}{n} = \frac{mc^2}{n}$; which, reduced, gives $x = \sqrt{\frac{mc^2}{n} + \frac{m^2b^2}{4n^2}} - \frac{mb}{2n}$.

The geometrical construction of this problem, from the equation $x^2 + \frac{mbx}{n} = \frac{mc^2}{n}$, may be as follows. In CB let there be taken CH : CB : : m : n, and let HK be drawn parallel to BA; then CH being $= \frac{mb}{n}$, and CK $= \frac{mc}{n}$, our equation will be changed to $x^2 + x \times CH = AC \times CK$, or to $CD \times \overline{CD + CH} = AC \times CK$. Upon CH as a diameter let the circle CTHQ be described, in which inscribe CG = AK; and in CG, produced, take CS = CA; and from S, through the center O, draw the right line STOQ, cutting the circumference in T and Q, and make CD = ST; then will D be the point required. For CG being = AK, and CS = CA; therefore will $AC \times CK = CS \times GS = ST \times SQ$ (*Euc.* 37. 3.) $= ST \times \overline{ST + TQ} = CD \times \overline{CD + CH}$; the very same as above.

PROBLEM XXI.

Having the perimeter of a right-angled triangle ABC, *and three perpendiculars* DE, DF, *and* DG, *falling from a point within the triangle upon the three sides thereof; to determine the sides.*

Suppose DA, DB, and DC to be drawn; and let DE $= a$, DF $= b$, DG $= c$, AB $= x$, BC $= y$, AC $= z$,



and the perimeter, AB + BC + AC, $= p$; then, the area of ADB being expounded by $\frac{ax}{2}$; that of BDC by $\frac{by}{2}$; that of ADC by $\frac{cz}{2}$; and that of the whole, ABC, by $\frac{xy}{2}$; we therefore have $\frac{ax}{2} + \frac{by}{2} + \frac{cz}{2} = \frac{xy}{2}$, or $ax + by + cz = xy$: moreover, we have $x^2 + y^2 = z^2$, and $x + y + z = p$, by the conditions of the problem. Let z be transposed in the last equation, and both sides squared, so shall $x^2 + 2xy + y^2 = p^2 - 2pz + z^2$, from which, if $x^2 + y^2 = z^2$ be subtracted, there will remain $2xy = p^2 - 2pz = 2ax + 2by + 2cz$ (*by the first equation*): whence $ax + by + \overline{c + p} \times z = \frac{1}{2}pp$: from this last equation subtract a times $x + y + z = p$, and there will remain $by - ay + \overline{p + c - a} \times z = \frac{1}{2}p^2 - ap$; also, if, from the same equation, b times $x + y + z = p$ be subtracted, there will remain $ax - bx + \overline{p + c - b} \times z = \frac{1}{2}p^2 - bp$; which two last equations, by putting $d = b - a$, $e = p + c - a$, $f = \frac{1}{2}p^2 - ap$, $g = p + c - b$, and $h = \frac{1}{2}p^2 - bp$, will stand thus: $dy + ez = f$, and $-dx + gz = h$; whence $y = \frac{f - ez}{d}$, and $x = \frac{gz - h}{d}$. Let these values of x and y be substituted in $x^2 + y^2 = z^2$, and we shall have $\frac{f^2 - 2efz + e^2z^2}{d^2} + \frac{g^2z^2 - 2ghz + h^2}{d^2} = z^2$, or $\overline{e^3 + g^2 - d^2} \times z^2 - \overline{2ef + 2gh} \times z = -f^2 - h^2$: put $e^2 + g^2 - d^2 = k$, $ef + gh = l$, and $f^2 + h^2 = m$; so shall $kz^2 - 2lz = -m$; whence $z^2 - \frac{2lz}{k} = -\frac{m}{k}$, and $z = \frac{l}{k}$

$\pm \sqrt{\frac{l^2}{k^2} - \frac{m}{k}} = \frac{l \pm \sqrt{l^2 - km}}{k}$: from which x ($= \frac{gz - h}{d}$) and y ($= \frac{f - ez}{d}$) will also be known.

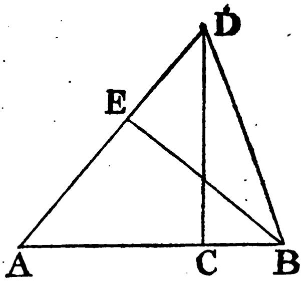
If a, b, and c be all equal to each other, the point D will be the center, and each of the given perpendiculars a radius of the inscribed circle; and the value of z, in this case, will be barely equal to $\frac{1}{2}p - a$; for the equation, $by - ay + p + c - a \times z = \frac{1}{2}p^2 - ap$, above found, here becomes $pz = \frac{1}{2}p^2 - ap$.

But, if only a and b (or DE and DF) be equal, then the equation will become $p + c - a \times z = \frac{1}{2}p^2 - ap$; and therefore $z = \frac{\frac{1}{2}p^2 - ap}{p + c - a}$; in which, if c be taken $= 0$, z will be $= \frac{\frac{1}{2}p^2 - ap}{p - a}$; where a is the side of the inscribed square.

PROBLEM XXII.

The perpendicular CD, *the difference of the sides*, AD — BD, *and the vertical angle* D, *of any plane triangle* ABD, *being given; to determine the sides.*

From B, upon AD (produced, if need be), let fall the perpendicular BE: let the sine of the angle BDE $= s$, its co-sine $= c$ (the radius being unity); also let the perpendicular CD $= p$, the lesser side BD $= x$, and the greater DA $= x + d$: then (*by prop.* 9, *in trigonometry*) as $1 : s :: x : sx =$ BE; and as $1 : c :: x : cx =$ ED. Now AB², being $=$ AD² + DB² — AD × 2DE (*Euc.* 13. 2.), will be expounded by $\overline{x + d}|^2 + x^2 - \overline{x + d} \times 2cx$, or $2x^2 + 2dx + d^2 - 2cx^2 - 2cdx$; whence, by reason of the



similar triangles ABE and ADC, it will be, as $2x^2 + 2dx + d^2 - 2cx^2 - 2cdx$ (AB^2) : s^2x^2 (BE^2) : : $x^2 + 2xd + dd$ (AD^2) : p^2 (DC^2); and, consequently, by multiplying extremes and means, $s^2x^4 + 2s^2dx^3 + s^2d^2x^2 = 2p^2\,{}^2 + 2p^2dx + p^2d^2 - 2p^2cx^2 - 2p^2cdx$; from whence, by transposition and division, we have $x^4 + 2dx^3 + \overline{d^2 - \frac{2p^2}{s^2} + \frac{2p^2c}{s^2}} \times x^2 + \overline{\frac{2p^2cd}{s^2} - \frac{2p^2d}{s^2}} \times x - \frac{p^2d^2}{s^2} = 0$. Which equation answering the conditions of the second case of biquadratics, explained at p. 154, we shall therefore have $x^2 + dx + \frac{p^2c - p^2}{s^2} \pm \sqrt{\frac{p^2d^2}{s^2} + \frac{\overline{p^2c - p^2}|^2}{s^4}}$; and consequently $x = -\frac{1}{2}d + \sqrt{\frac{d^2}{4} + \frac{p^2 - p^2c}{s^2} + \sqrt{\frac{p^2d^2}{s^2} + \frac{\overline{p^2c - p^2}|^2}{s^4}}}$.

Otherwise.

Supposing s, c, and p to be the same as before, put half the given difference of the sides $= a$, and half their sum $= x$; then the greater side AD will be $= x + a$, and the lesser BD $= \overline{x - a}$; wherefore (*by trigonometry*) $1 : s :: \overline{x - a} : s \times \overline{x - a} = BE$; and $1 : c :: \overline{x - a} : c \times \overline{x - a} = DE$: but AB^2 is $= \overline{AD^2 + DB^2} - \overline{2DE \times AD} = \overline{x + a}|^2 + \overline{x - a}|^2 - 2c \times \overline{x - a} \times \overline{x + a} = 2x^2 + 2a^2 - 2cx^2 + 2ca^2$; whence, by reason of the similar triangles ABE, ADC, it will be $2x^2 + 2a^2 - 2c\,{}^2 + 2ca^2$ (AB^2) : $s^2 \times \overline{x - a}|^2$ (BE^2) : : $\overline{x + a}|^2$ (AD^2) : p^2 (DC^2); and consequently $s^2 \times \overline{x - a}|^2 \times \overline{x + a}|^2 = \overline{2x^2 + 2a^2 - 2cx^2 + 2ca^2} \times p^2$, or $s^2x^4 - 2s^2a^2x^2 + s^2a^4 = 2p^2x^2 - 2cp^2x^2 + 2p^2a^2 + 2cp^2a^2$; whence, by transposition and division, $x^4 - 2a^2x^2 - \frac{2p^2x^2}{s^2} + \frac{2cp^2x^2}{s^2} = \frac{2p^2a^2}{s^2} + \frac{2cp^2a^2}{s^2} - a^4$. Substitute

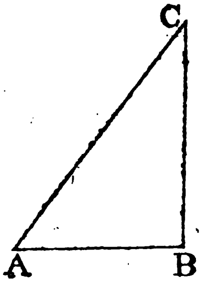
$f = a^2 + \frac{p^2}{s^2} - \frac{cp^2}{s^2}$, and $g = a^2 \times \overline{\frac{2p^2 + 2cp^2}{s^2} - a^2}$; then the equation will stand thus, $x^4 - 2fx^2 = g$: whence x is found $= \sqrt{f \pm \sqrt{f^2 + g}}$.

If, instead of the difference, the sum of the sides had been given, in order to find the difference, the method of operation would have been the very same; only, instead of finding the value of x in terms of a, by means of the equation $s^2x^4 - 2s^2a^2x^2 + s^2a^4 = 2p^2x^2 - 2cp^2x^2 + 2p^2a^2 + 2cp^2a^2$, that of a must have been found, in terms of x, from the same equation.

PROBLEM XXIII.

Having one leg AB *of a right-angled triangle* ABC; *to find the other leg* BC, *such that the rectangle under their difference* (BC — AB) *and the hypothenuse* AC *may be equal to the area of the triangle.*

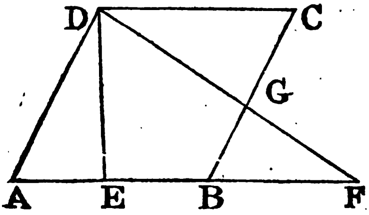
Put AB $= a$, and BC $= x$; so shall AC $= \sqrt{aa + xx}$; and $\frac{ax}{2} = \overline{x - a} \,.\, \sqrt{aa + xx}$, by the conditions of the problem. By squaring both sides of this equation, we have $\frac{1}{4}a^2x^2 = \overline{x^2 - 2ax + a^2} \times \overline{aa + xx}$: in which the quantities x and a being concerned exactly alike, the solution will therefore be brought out from the general method for extracting the roots of these kinds of equations (delivered at p. 156): according to which, having divided the whole by a^2x^2, we get $\frac{1}{4} = \overline{\frac{x}{a} - 2 + \frac{a}{x}} \times \overline{\frac{a}{x} + \frac{x}{a}}$; which, by making $z = \frac{x}{a} + \frac{a}{x}$, will be reduced down to $\frac{1}{4} = z - \overline{2 \times z}$, or $z^2 - 2z = \frac{1}{4}$: whence z is given $= 1 + \sqrt{\frac{5}{4}}$. But since $\frac{x}{a} + \frac{a}{x} = z$,



we have $x^2 - azx = -a^2$; and therefore $x = \frac{az}{2} + \sqrt{\frac{a^2z^2}{4} - a^2} = \frac{a}{2} \times \overline{z + \sqrt{zz - 4}}$: which, by substituting the value of z, becomes $x = \frac{a}{2} \times \overline{1 + \sqrt{\tfrac{5}{4}} + \sqrt{\sqrt{5} - \tfrac{7}{4}}}$.

PROBLEM XXIV.

To draw a right line DF *from one angle* D *of a given rhombus* ABCD, *so that the part thereof* FG, *intercepted by one of the sides including the opposite angle and the other side produced, may be of a given length.*

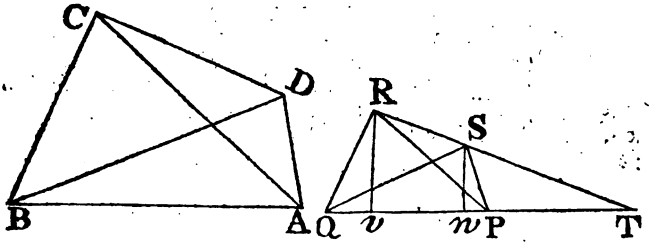


Let DE be perpendicular to AB; and let AB ($=$ AD) $= a$, AE $= b$, FG $= c$, and AF $= x$: then DF^2 ($=$ AF^2 $+$AD^2$-$2AE$\times$AF) $= xx + aa - 2bx$; and, by similar triangles, $xx + aa - 2bx$(DF^2)$:xx$(AF^2) $::cc$ (FG^2) $: \overline{x - a}|^2$ (BF^2); and consequently $\overline{xx + aa - 2bx} \times \overline{xx - 2ax + aa} = ccxx$. Make $ma = b$, and $na = c$; so shall our equation become $\overline{xx + aa - 2max} \times \overline{xx - 2ax + aa} = n^2a^2x^2$: which, divided by a^2x^2, gives $\overline{\frac{x}{a} + \frac{a}{x} - 2m} \times \overline{\frac{x}{a} + \frac{a}{x} - 2} = n^2$: this, by making $z = \frac{x}{a} + \frac{a}{x}$, becomes $\overline{z - 2m} \times \overline{z - 2} = n^2$: therefore $z^2 - \overline{2m + 2} \times z = n^2 - 4m$, and $z = 1 + m + \sqrt{n^2 + \overline{1 - m}|^2} = \frac{a + b + \sqrt{c^2 + \overline{a - b}|^2}}{a}$, by restoring the values m and n. From whence the value of x will be also known; for $\frac{x}{a} + \frac{a}{x}$ being $= z$, we have, by reduction, $x^2 - azx = -aa$; and therefore $x = \frac{a}{2} \times \overline{z + \sqrt{zz - 4}}$.

PROBLEM XXV.

The diagonals AC, BD, *and all the angles* DAB, ABC, BCD, *and* CDA, *of a trapezium* ABCD, *being given; to determine the sides.*

Let PQRS be another trapezium similar to ABCD, whose side PQ is unity; and let QP and RS be produced till they meet in T: also let PR and QS be drawn, and make Rv and Sw perpendicular to TQ. Let the



(natural) sine of the given angle STP, to the radius 1, be put $= m$; that of TSP, or PSR, $= n$; that of TRQ $= p$; the co-sine of SPQ $= r$; that of RQv $= s$; AC $= a$; BD $= b$; and PT $= x$. Then (*by plane trigonometry*) $n : m :: x : \text{PS} = \frac{mx}{n}$; and $1 : \frac{mx}{n}$ (PS) $:: r : \text{P}w = \frac{rmx}{n}$; whence (*by Euc.* 13. 2.) QS^2 ($= \text{QP}^2 + \text{PS}^2 - 2\text{PQ} \times \text{P}w) = 1 + \frac{m^2x^2}{nn} - \frac{2rmx}{n}$.

Again (*by trigonometry*), $p : m :: 1 + x$ (TQ) : $\text{QR} = \frac{m + mx}{p}$; and $1 : s :: \frac{m + mx}{p}$ (QR) : $\text{Q}v = \frac{ms + msx}{p}$. And therefore PR^2 ($= \text{PQ}^2 + \text{QR}^2 -$

$2PQ \times Qv) = 1 + \frac{\overline{m + mx}|^2}{pp} - \frac{2ms + 2msx}{p}$. But because of the similar figures ABCD, PQRS, it will be, $AC^2 : BD^2 :: PR^2 : QS^2$, that is, $a^2 : b^2 ::$ $1 + \frac{\overline{m + mx}|^2}{pp} - \frac{2ms + 2msx}{p} : 1 + \frac{m^2x^2}{nn} - \frac{2rmx}{n}$; and consequently $a^2 + \frac{a^2m^2x^2}{nn} - \frac{2a^2rmx}{n} = b^2 + \frac{b^2m^2}{pp}$ $+ \frac{2b^2m^2x^2}{pp} + \frac{b^2m^2x^2}{pp} - \frac{2b^2ms}{p} - \frac{2b^2msx}{p}$: whence, writing $f = \frac{a^2m^2}{nn} - \frac{b^2m^2}{pp}$, $g = \frac{bbms}{p} - \frac{bbm^2}{pp} - \frac{aarm}{n}$, and $h = b^2 - a^2 + \frac{b^2m^2}{pp} - \frac{2bbms}{p}$, we have $fx^2 + 2gx = h$: which, solved, gives $x = \sqrt{\frac{h}{f} + \frac{gg}{ff}} - \frac{g}{f}$: from whence SQ will also be known: and again, by reason of the similar figures, it will be as $QS : QP$ (unity) $:: BD : AB$; which, therefore, is likewise known: from whence the rest of the sides BC, CD, and DA will all be found by plane trigonometry.

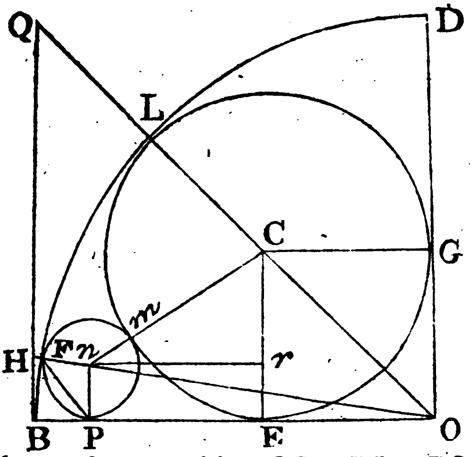
The last problem is indeterminate in that particular case, where the trapezium may be inscribed in a circle, or where the sum of the two opposite angles is equal to two right ones; for then there can but one diagonal be given in the question, because the value of the other depends entirely upon that.

PROBLEM XXVI.

Supposing BOD *to be a quadrant of a given circle; to find the semi-diameter* CE, *or* CL, *of the circle* CEGL, *inscribed therein; and likewise the semi-diameter of the little circle* $nFmP$, *touching both the other circles* DLB, LME, *and also the right line* OB.

Let BQ, Pn, and CE be perpendicular to BO; join C, n, and O, n; and draw OC meeting BQ in Q, and nr parallel to BO, meeting CE in r: put OB

(= BQ) = 1, OQ (= $\sqrt{2}$, *by Euc.* 47. 1.) = b, and Pn (= nm) = x. Then, by reason of the similar triangles OBQ, OEC, it will be, OQ : BQ : : OC :



CE: whence, *by composition*, OQ + BQ : BQ : : OL (OC + CE) : CE; that is, $b + 1 : 1 : : 1 :$ CE = $\frac{1}{b+1}$ $\left(= \frac{b-1}{\overline{b+1} \times \overline{b-1}}\right) = \frac{b-1}{b^2-1} = b - 1 = \sqrt{2} - 1$; which let be denoted by a, then we shall have Cn + Cr = $2a$, and Cn — Cr = $2x$; and therefore nr $\left(= \sqrt{\overline{\mathrm{C}n + \mathrm{C}r} \times \overline{\mathrm{C}n - \mathrm{C}r}}\right) = 2\sqrt{ax}$. Moreover, O$n$ + Pn being = 1, and On — Pn = $1 - 2x$; thence will OP = $\sqrt{1-2x}$; which, also, being = PE + OE ($2\sqrt{ax} + a$), we therefore have $\sqrt{1-2x} = 2\sqrt{ax} + a$; whereof both sides being squared, there arises $1 - 2x = 4ax + 4a\sqrt{ax} + a^2$, or $1 - a^2 - 2x - 4ax = 4a\sqrt{ax}$; which, because $1 - aa$ is $2a$, will be $a - \overline{1 + 2a} \times x = 2a\sqrt{ax}$: this, squared, gives $a^2 - \overline{1 + 2a} \times 2ax + \overline{1 + 2a}\big]^2 \times x^2 = 4a^3x$; whence $\overline{1 + 2a}\big]^2 \times x^2 - \overline{1 + 2a} \times 2ax - 4a^3x = -aa$; which, by writing

$b - 1$ instead of its equal a, becomes $\overline{2b - 1}|^2 \times x^2 - \overline{7b - 9} \times 2x = 2b - 3$; therefore $x^2 - \frac{7b - 9}{\overline{2b - 1}|^2} \times 2x = \frac{2b - 3}{\overline{2b - 1}|^2}$; from whence x is found $=$

$$\frac{7b - 9}{\overline{2b - 1}|^2} \pm \sqrt{\frac{2b - 3}{\overline{2b - 1}|^2} + \frac{\overline{7b - 9}|^2}{\overline{2b - 1}|^4}} =$$

$$\frac{7b - 9 \pm \sqrt{\overline{2b - 3} \times \overline{2b - 1}|^2 + \overline{7b - 9}|^2}}{\overline{2b - 1}|^2} =$$

$\frac{7b - 9 \pm \sqrt{8b^3 + 29b^2 - 112b + 78}}{\overline{2b - 1}|^2}$: which, by writing $\sqrt{2}$ for b, becomes $\frac{7\sqrt{2} - 9 \pm \sqrt{136 - 96\sqrt{2}}}{\overline{2\sqrt{2} - 1}|^2} =$ $\frac{7\sqrt{2} - 9 \pm 6\sqrt{2} \mp 8}{\overline{2\sqrt{2} - 1}|^2}$, that is, equal to $\frac{13\sqrt{2} - 17}{\overline{2\sqrt{2} - 1}|^2}$, or to $\frac{\sqrt{2} - 1}{\overline{2\sqrt{2} - 1}|^2}$; which last is the root required, the other being manifestly too large: but this value will be reduced to $\frac{5\sqrt{2} - 1}{49}$. Therefore OP $(= \sqrt{1 - 2x})$ is given $= \sqrt{\frac{51 - 10\sqrt{2}}{49}} = \frac{5\sqrt{2} - 1}{7} = 7Pn$; and consequently BH $= \frac{1}{7}$BQ; from whence we have the following construction.

In the tangent BQ take BH $= \frac{1}{7}$BO; draw HO, cutting the circumference BDL in F, and make the angle OFP $= \frac{1}{2}$OHB, and draw PN parallel to BQ, meeting OH in n, the center of the lesser circle required.

SCHOLIUM.

In the preceding solution it was required, not only to extract the square root of the radical quantities 136 —

$96\sqrt{2}$ and $51-10\sqrt{2}$, but likewise to take away the radical quantity from the denominator of the fraction $\frac{\sqrt{2}-1}{\overline{2\sqrt{2}-1}\,\rceil^{2}}$, and confine it, wholly, to the numerator: all which being somewhat difficult (and, for that reason, omitted in the introduction, as too discouraging to a young beginner), I shall therefore take the opportunity to explain here the manner of proceeding in such like cases, when they happen to occur.

First, then, with regard to the extraction of roots out of radical quantities, let there be proposed $A \pm \sqrt{B}$, A being the rational, and $\sqrt{B}$ the irrational part thereof; and let the root required be represented by $\sqrt{x} \pm \sqrt{y}$; the square of which will be $x+y \pm 2\sqrt{xy}$, or $x+y \pm \sqrt{4xy}$; therefore we have $x+y \pm \sqrt{4xy} = A \pm \sqrt{B}$. Let the irrational, as well as the rational parts of these two equal quantities be now compared together; so shall $x+y=A$, and $\sqrt{4xy}=\sqrt{B}$: from the square of the former of which equations subtract that of the latter; whence will be had $xx-2xy+yy=A^2-B$; and, by taking the square root, $x-y=\sqrt{A^2-B}$; which added to, and subtracted from $x+y=A$, &c. gives $x=\frac{A+\sqrt{A^2-B}}{2}$, and $y=\frac{A-\sqrt{A^2-B}}{2}$.

In the first of the two cases above specified, the quantity whose square root is to be extracted being $136-96\sqrt{2}$, we have $A=136$, and $B=18432$ $(=\overline{96}\rceil^{2}\times 2)$; whence we have x $\left(=\frac{A+\sqrt{A^2-B}}{2}\right)=72$, and y $\left(=\frac{A-\sqrt{A^2-B}}{2}\right)=64$; and consequently $\sqrt{x}-\sqrt{y}=\sqrt{72}-\sqrt{64}=6\sqrt{2}-8$, the required square root of $136-96\sqrt{2}$. After the very same manner, the square root of the other radical quantity $51-10\sqrt{2}$, or $51-\sqrt{200}$ will be found to be $5\sqrt{2}-1$:

for, A being here $= 51$, and $B = 200$, we have $x = 50$, and $y = 1$; and consequently $\sqrt{x} - \sqrt{y} = 5\sqrt{2} - 1$.

What has been said, thus far, relates to the extraction of the square root only; but the same method is easily extended to the cube, biquadratic, or any other root. Let us take an instance thereof in the cube root; where we will suppose the given quantity, out of which the root is to be extracted, to be represented by $A \pm \sqrt{B}$, as before. Then, if the rational part of the root be denoted by x, and the irrational part by $\sqrt{y}$; the root itself will be expressed by $x \pm \sqrt{y}$; and its cube by $x^3 \pm 3x^2\sqrt{y} + 3xy \pm y\sqrt{y}$: from whence, by proceeding as in the extraction of the square root, we have $x^3 + 3xy = A$, and $3x^2\sqrt{y} + y\sqrt{y} = \sqrt{B}$. Let the sum and the difference of these two equations be taken, and there will come out $x^3 + 3x^2\sqrt{y} + 3xy + y\sqrt{y} = A + \sqrt{B}$, and $x^3 - 3x^2\sqrt{y} + 3xy - y\sqrt{y} = A - \sqrt{B}$; whereof the cube root being extracted on both sides, we thence have $x + \sqrt{y} = \overline{A + \sqrt{B}}\rceil^{\frac{1}{3}}$, and $x - \sqrt{y} = \overline{A - \sqrt{B}}\rceil^{\frac{1}{3}}$: let the two last equations be added together, and the sum be divided by 2: so shall $x = \dfrac{\overline{A + \sqrt{B}}\rceil^{\frac{1}{3}} + \overline{A - \sqrt{B}}\rceil^{\frac{1}{3}}}{2}$; and, by multiplying the same equations together, we get $x^2 - y = \overline{A^2 - B}\rceil^{\frac{1}{3}}$, and consequently $y = x^2 - \overline{A^2 - B}\rceil^{\frac{1}{3}}$; whence y is likewise known.

Universally, let the index of the root to be extracted be denoted by n, and let the root itself be represented by $x \pm \sqrt{y}$ (as above). Then this expression raised to the nth power, will be $x^n \pm nx^{n-1}\sqrt{y} + n \times \frac{n-1}{2}x^{n-2}y \pm n \times \frac{n-1}{2} \times \frac{n-2}{3}x^{n-3}y\sqrt{y}$, &c. from whence, still following the same method, we shall here get $x^n + n \times \frac{n-1}{2}x^{n-2}y$, &c. $= A$, and

$nx^{n-1}\sqrt{y} + n \times \frac{n-1}{2} \times \frac{n-2}{3}\, x^{n-2}y\sqrt{y}$, &c. $= \sqrt{B}$: let, therefore, the root of the sum, and also of the difference of these two last equations, be taken, and you will have $x + \sqrt{y} = \overline{A + \sqrt{B}}\rceil^{\frac{1}{n}}$, and $x \backsim \sqrt{y} = \overline{A - \sqrt{B}}\rceil^{\frac{1}{n}}$; which two equations being added together, x will be found $= \frac{\overline{A + \sqrt{B}}\rceil^{\frac{1}{n}} \pm \overline{A - \sqrt{B}}\rceil^{\frac{1}{n}}}{2}$ $= \frac{\overline{A + \sqrt{B}}\rceil^{\frac{1}{n}}}{2} \pm \frac{\overline{A^2 - B}\rceil^{\frac{1}{n}}}{2 \times \overline{A + \sqrt{B}}\rceil^{\frac{1}{n}}}$: and, if the same equations be multiplied together, you will have $x^2 \backsim y = \overline{A^2 - B}\rceil^{\frac{1}{n}}$; whence $y = x^2 \pm \overline{A^2 - B}\rceil^{\frac{1}{n}}$: the use of which conclusions will appear by the following examples.

First, let it be proposed to extract the cube root of the radical quantity $26 + 15\sqrt{3}$, or $26 + \sqrt{675}$. Here, A being $= 26$, $B = 675$, and $n = 3$, we have $x \left(= \frac{\overline{26 + \sqrt{675}}\rceil^{\frac{1}{3}}}{2} \pm \frac{1}{2 \times \overline{26 + \sqrt{675}}\rceil^{\frac{1}{3}}} = \frac{3{,}732 \pm {,}268}{2}\right)$ $= 2$; and $y \left(= 4 - \overline{676 - 675}\rceil^{\frac{1}{3}}\right) = 3$; and consequently $x + \sqrt{y} = 2 + \sqrt{3} =$ the value required for $\overline{2 + \sqrt{3}} \times \overline{2 + \sqrt{3}} \times \overline{2 + \sqrt{3}} = 26 + \sqrt{675}$.

Again, let it be required to extract the biquadratic root of $161 + \sqrt{25920}$. In this case, A being $= 161$, $B = 25920$, and $n = 4$, we have $x \left(= \frac{\overline{321{,}996896}\rceil^{\frac{1}{4}}}{2} \pm \frac{1}{2 \times \overline{321{,}99, \text{\&c.}}\rceil^{\frac{1}{4}}} = \frac{4{,}236 \pm {,}236}{2}\right) = 2$, and

$y\ (= 4 + \overline{25921 - 25920}\rceil^{\frac{1}{4}}) = 5$; therefore the root sought is, here, $= 2 + \sqrt{5}$.

Lastly, if it were required to find the first sursolid root of $76 + \sqrt{5808}$; then, by proceeding in the same manner, x will be found $(= \frac{2{,}732 \mp {,}732}{2}) = 1$, and $y\ (= 1 - \overline{5776 - 5808}\rceil^{\frac{1}{5}}) = 3$: and so of others. But it is to be observed, that the second part of the value of x, to which both the signs + and — are prefixed, is to be taken affirmative or negative, according as that or this shall be found requisite to make the value of x come out a whole, or rational number: and that, if neither of the signs give such a value of x, then this method is of no use, and we may safely conclude that the quantity proposed does not admit of such a root as we would find. It may also be proper to remark here, that, if the upper sign in the value of x be taken, the upper sign in that of y must be taken accordingly; and that the application of logarithms will be of use to facilitate the extraction of the root $\overline{A + \sqrt{B}}\rceil^{\frac{1}{n}}$, as being sufficiently exact to determine whether x be a whole number, and, if so, what it is.

Thus much in relation to the extraction of the roots of radical quantities; it remains now to explain the manner of taking away radical quantities out of the denominator of a fraction, and transplanting them into the numerator.

In order to which, supposing r to denote a whole number, it is evident, in the first place, that

$x^r - y^r = \overline{x - y} \times \overline{x^{r-1} + x^{r-2}y + x^{r-3}y^2 \ldots + y^{r-1}}$;

since, by an actual multiplication, the product appears to be $\left\{ \begin{array}{l} x^r + x^{r-1}y + x^{r-2}y^2 + x^{r-3}y^3, \text{\&c.} \\ \quad - x^{r-1}y - x^{r-2}y^2 - x^{r-3}y^3 \ldots - y^r \end{array} \right\}$, where all the terms, except the first and last, destroy one another. Hence, by equal division, we have

$\frac{1}{x-y}=\frac{x^{r-1}+x^{r-2}y+x^{r-3}y^2\ldots+y^{r-1}}{x^r-y^r}$. And, in the very same manner, it will appear that $\frac{1}{x+y}=\frac{x^{r-1}-x^{r-2}y+x^{r-3}y^2-x^{r-4}y^3\ldots\pm y^{r-1}}{x^r\pm y^r}$: where the sign — or +, in the denominator, takes place, according as the number r is even or odd.

Let now $x=\mathrm{A}^m$, and $y=\mathrm{B}^n$; then our equations will become

$$\frac{1}{\mathrm{A}^m-\mathrm{b}^n}=\frac{\mathrm{A}^{rm-m}+\mathrm{A}^{rm-2m}\mathrm{B}^n+\mathrm{A}^{rm-3m}\mathrm{B}^{2n}\ldots\ldots+\mathrm{B}^{rn-n}}{\mathrm{A}^{rm}-\mathrm{B}^{rn}}$$

$$\frac{1}{\mathrm{A}^m+\mathrm{B}^n}=\frac{\mathrm{A}^{rm-m}-\mathrm{A}^{rm-2m}\mathrm{B}^n+\mathrm{A}^{rm-3m}\mathrm{B}^{2n}\ldots\ldots\pm\mathrm{B}^{rn-n}}{\mathrm{A}^{rm}+\mathrm{B}^{rn}}.$$

From which theorems, or general *formulæ*, the matter proposed to be done may be effected with great facility: for, supposing $\frac{1}{\mathrm{A}^m-\mathrm{B}^n}$, or $\frac{1}{\mathrm{A}^m+\mathrm{B}^n}$ to be a fraction having radical quantities A^m, B^n in the denominator, it is plain, that its equal value, given by the said equations, will have its denominator entirely free from radical quantities, if r be so assumed that both rm and rn may be integers.

To exemplify which, let the fraction $\frac{1}{\sqrt{2}-1}$, or $\frac{1}{\overline{2}\rceil^{\frac{1}{2}}-1}$ be propounded; then, A being $=2$, $\mathrm{B}=1$, $m=\frac{1}{2}$, and $n=1$, we shall, by taking $r=2$, have (*from theorem* 1) $\frac{1}{\overline{2}\rceil^{\frac{1}{2}}-1}=\frac{\overline{2}\rceil^{\frac{1}{2}}+1}{2-1}=\sqrt{2}+1$.

Again, let the given fraction be $\frac{1}{\sqrt{cx} + \sqrt[4]{c^4 + x^4}}$ or $\frac{1}{\overline{cx}\,]^{\frac{1}{2}} + \overline{c^4 + x^4}\,]^{\frac{1}{4}}}$. In which case, A being $= cx$, $B = c^4 + x^4$, $m = \frac{1}{2}$, and $n = \frac{1}{4}$, we shall, by taking $r = 4$, have $\frac{1}{\overline{cx}\,]^{\frac{1}{2}} + \overline{c^4 + x^4}\,]^{\frac{1}{4}}} =$

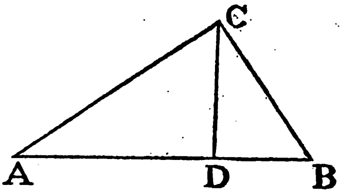
$$\frac{\overline{cx}\,]^{\frac{3}{2}} - cx \times \overline{c^4 + x^4}\,]^{\frac{1}{4}} + cx^{\frac{1}{2}} \times \overline{c^4 + x^4}\,]^{\frac{1}{2}} - \overline{c^4 + x^4}\,]^{\frac{3}{4}}}{c^2x^2 - c^4 - x^4}.$$

If the numerator be not a unit, you may proceed in the same manner, and multiply afterwards by the numerator given. Thus, in the case mentioned at the beginning of this scholium, we had given $\frac{\sqrt{2} - 1}{\overline{2\sqrt{2} - 1}\,]^2}$, which may be reduced to $\overline{\sqrt{2} - 1} \times \frac{1}{2\sqrt{2} - 1} \times \frac{1}{2\sqrt{2} - 1}$, or to $\overline{\sqrt{2} - 1} \times \frac{1}{8^{\frac{1}{2}} - 1} \times \frac{1}{8^{\frac{1}{2}} - 1}$: but $\frac{1}{8^{\frac{1}{2}} - 1}$ is found (*by theorem* 1) to be $= \frac{\sqrt{8} + 1}{8 - 1} = \frac{2\sqrt{2} + 1}{7}$: whence our expression becomes $\overline{\sqrt{2} - 1} \times \frac{2\sqrt{2} + 1}{7} \times \frac{2\sqrt{2} + 1}{7}$: which, by multiplication, &c. is reduced, at length, to $\frac{5\sqrt{2} - 1}{49}$.

PROBLEM XXVII.

Having one leg, AC, *of a right-angled triangle* ABC, *to find the other leg* BC, *so that the hypothenuse* AB *shall be a mean proportional between the perpendicular* CD *falling thereon, and the perimeter of the triangle.*

Put AC = a, and BC = x; then will AB = $\sqrt{xx+aa}$, and CD $\left(=\frac{AC \times BC}{AB}\right) = \frac{ax}{\sqrt{xx+aa}}$: therefore, *by the question*, $x+a+\sqrt{xx+aa} : \sqrt{xx+aa} :: \sqrt{xx+aa} : \frac{ax}{\sqrt{xx+aa}}$; and consequently $\frac{ax^2+a^2x}{\sqrt{xx+aa}} + ax = xx+aa$: whence $a^2x^4 + 2a^3x^3 + a^4x^2 = \overline{xx+aa-ax}\big|^2 \times \overline{xx+aa}$. Divide by a^3x^3 (according to the rule at page 156), so shall $\frac{x}{a} + 2 + \frac{a}{x} = \overline{\frac{x}{a} + \frac{a}{x} - 1}\big|^2 \times \overline{\frac{x}{a} + \frac{a}{x}}$: which, by making $z = \frac{x}{a} + \frac{a}{x}$, is reduced to $z + 2 = \overline{z-1}\big|^2 \times z$, or $z^3 - 2z^2 = 2$. This, solved (by the rule for cubics), gives $z = \frac{2}{3} + \frac{1}{3} \times \overline{35+\sqrt{1161}}\big|^{\frac{1}{3}} + \frac{1}{3} \times \frac{4}{\overline{35+\sqrt{1161}}\big|^{\frac{1}{3}}} = 2{,}359304$: whence $x \left(= \frac{a}{2} \times \overline{z \pm \sqrt{zz-4}}\right)$ will likewise be known.

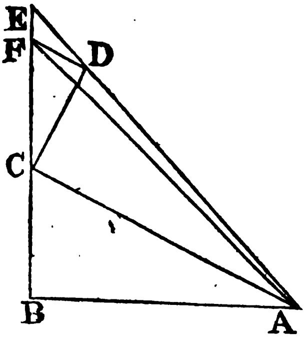


PROBLEM XXVIII.

The base AB, *and the perpendicular* BE, *of a right-angled triangle* ABE, *being given; it is proposed to find a point* (C) *in the perpendicular, from whence two right lines* CA *and* CD *being drawn, at right angles to each other, the two triangles* ACD *and* ABC, *formed from thence, shall be equal.*

Suppose DF parallel to AC, and let AF be drawn: putting AB = a, BE = b, BC = x, and AC $(\sqrt{a^2+x^2})$

$= y$. Then, since FD is parallel to AC, the triangle ACF will be equal to ACD, or ABC; and therefore CF $=$ BC $= x$; whence we have EF ($=$ EB $-$ BF) $= b - 2x$, and EC ($=$ EB $-$ BC) $= b - x$: moreover, by reason of the similar triangles ABC and CDF, we have y (AC) : x (BC) : : x (CF) : FD $= \frac{xx}{y}$.



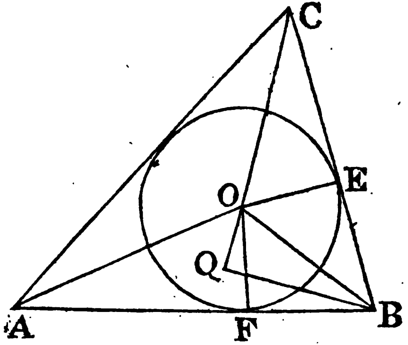
Whence, because of the parallel lines AC and FD, it will be, $\frac{x^2}{y}$ (FD) : $b - 2x$ (EF) : : y (AC) : $b - x$ (CE); and consequently $\frac{b\;{}^2 - x^3}{y} = \overline{b - 2x} \times y$, or $bx^2 - x^3 = \overline{b - 2x} \times y^2$; which equation, by writing $a^2 + x^2$, instead of its equal, y^2, becomes $bx^2 - x^3 = ba^2 + bx^2 - 2a^2x - 2x^3$; whence we have $x^3 + 2a^2x = ba^2$, and therefore $x =$

$$\sqrt[3]{\frac{ba^2}{2} + a^2\sqrt{\frac{b^2}{4} + \frac{8a^2}{27}}} - \frac{\frac{2}{3}aa}{\sqrt[3]{\frac{baa}{2} + a^2\sqrt{\frac{bb}{4} + \frac{8aa}{27}}}}.$$

PROBLEM XXIX.

Three lines, AO, BO, CO, *drawn from the angular points of a triangle to the center of the inscribed circle, being given; to find the radius of the circle, and the sides of the triangle.*

If upon CQ produced, the perpendicular BQ be let fall, and the radii OE, OF be drawn to the points of contact, the triangles BOQ and AOF will appear to be equiangular; because all the angles of the triangle ABC being equal to two right ones, the sum of all their halves, OCB + OBC + OAF, will be equal to one right angle; but the two former of these, OCB + OBC, is equal to the external angle QOB; therefore QOB + OAF = a right angle = QOB + OBQ, and consequently OAF = OBQ. Put now AO $= a$, BO $= b$, CO $= c$, and OF $= x$: then, because of the similar triangles, we have $a : x :: b :$ OQ $= \frac{bx}{a}$; whence BQ^2 $= b^2 - \frac{bbxx}{aa}$, and BC^2 (= CO^2 + BO^2 + 2CO × OQ) $= b^2 + c^2 + \frac{2bcx}{a}$. But BC^2 : BQ^2 : : OC^2 : OE^2; that is, $\frac{ab^2 + ac^2 + 2bcx}{a} : \frac{a^2b^2 - b^2x^2}{aa} :: c^2 : x^2$. From whence we get this equation, *viz.* $ax^2 \times \overline{ab^2 + ac^2 + 2bcx} = b^2c^2 \times \overline{a^2 - x^2}$; which, by reduction, will become $x^3 + \overline{\frac{ab}{2c} + \frac{ac}{2b} + \frac{bc}{2a}} \times x^2 = \frac{abc}{2}$: whence x may be found, and from thence the sides of the triangle.



If two of the given lines, as OC and OB, be supposed equal, the result will be more simple: for, by writing b for c in the equation $ax^2 \times \overline{ab^2 + ac^2 + 2bcx} = b^2c^2 \times \overline{a^2 - x^2}$, &c. we shall have $ab^2x^2 \times \overline{2a + 2x} = b^4 \times \overline{a^2 - x^2}$; which, divided by $b^2 \times \overline{a + x}$, gives $2ax^2 = b^2 \times \overline{a - x}$; whence $x^2 + \frac{b^2x}{2a} = \frac{1}{2}b^2$, and

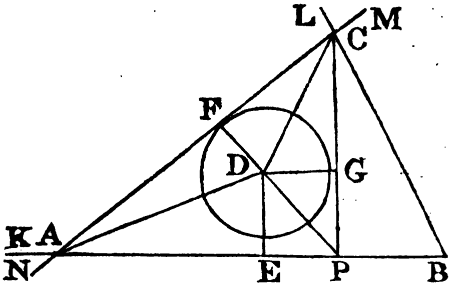
$$x = \sqrt{\frac{b^2}{2} + \frac{b^4}{16aa}} - \frac{b^2}{4a} = \frac{b\sqrt{8aa + bb} - bb}{4a}.$$

From the same equations the problem may be resolved, when the distances from the three angular points to the circumference of the inscribed circle are given: for, denoting the said distances by f, g, and h, you will have AO $= x + f$, BO $= x + g$, and CO $= x + h$; which values being written in the room of a, b, and c, there will arise an equation of six dimensions: by means whereof x may be found.

PROBLEM XXX.

To draw a line NM *to touch a circle* D, *given in magnitude and position, so that the part thereof* AC, *intercepted by two other lines* BK, BL, *given in position, shall be of a given length.*

Suppose CP and DE to be perpendicular to AB, and DF and DG to AC and PC, respectively; and let DA, DC, and DP be drawn; putting DE $= a$, DF $= b$, AC $= c$, BE $= d$, PC $= x$, PA $= y$, and the tangent of the given angle BCP to the radius $1 = t$. Then, *by trigonometry*, $1 : t :: x : tx =$ BP; therefore DG ($=$ PE) $= d - tx$; which, multiplied by $\frac{1}{2}$PC, or $\frac{x}{2}$, gives $\frac{dx - tx^2}{2}$, for the area of the triangle CDP: in like manner, the area of the triangle PDA will be found $= \frac{ay}{2}$; and that of ADC $= \frac{bc}{2}$; which three, added together, are equal to the whole area ACP; that is, $\frac{dx - tx^2}{2} + \frac{ay}{2} + \frac{bc}{2} = \frac{xy}{2}$; and conse-



quently $bc + dx - tx^2 = xy - ay$. Let both sides of this equation be squared, and you will have $\overline{bc + dx - tx^2}|^2 = \overline{x - a}|^2 \times y^2 = \overline{x - a}|^2 \times \overline{c^2 - x^2}$; that is, $b^2c^2 + 2bcdx - 2bctx^2 + d^2x^2 - 2dtx^3 + t^2x^4 = -x^4 + 2ax^3 - a^2x^2 + c^2x^2 - 2ac^2x + a^2c^2$; whence $\overline{1 + t^2} \times x^4 - \overline{2a + 2dt} \times x^3 + \overline{a^2 - c^2 + d^2 - 2bct} \times x^2 + \overline{2ac^2 + 2bcd} \times x - a^2c^2 + b^2c^2 = 0$: from which the value of x may be found; and then, the value of y $(= \sqrt{c^2 - x^2})$ being known, the position of the points A and C, through which the line must pass, will also become known.

If the given angle B be a right one, the point B will coincide with P; and, therefore, t in this case being $= 0$, the equation will become $x^4 - 2ax^3 + \overline{a^2 - c^2 + d^2} \times x^2 + \overline{2ac^2 + 2bcd} \times x - a^2c^2 + b^2c^2 = 0$.

When the circle touches the right line AB, a will then be equal to b: and, in that case, the equation will be $\overline{1 + t^2} \times x^4 - \overline{2a + 2dt} \times x^2 + \overline{a^2 - c^2 + d^2 - 2act} \times x + 2ac^2 + 2acd = 0$, because the two last terms $-a^2c^2 + b^2c^2$ destroying each other, the whole may, here, be divided by x.

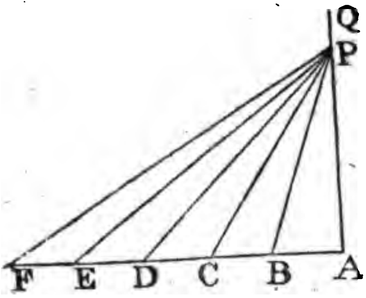
Lastly, if b be $= 0$, or the line AC, instead of touching a circle, be required to pass through a given point, the equation will then become $\overline{1 + t^2} \times x^4 - \overline{2a + 2dt} \times x^3 + \overline{a^2 - c^2 + d^2} \times x^2 + 2ac^2x - a^2c^2 = 0$.

PROBLEM XXXI.

Supposing AQ *perpendicular to* AF, *and the given right line* AF (50) *to be divided into five equal parts, in the points* B, C, D, *and* E; *to find a point* P *in the perpendicular* AQ, *from which, if five right lines be drawn to the points* B, C, D, E, *and* F, *the sum of the outermost* PF + PE *shall be equal to the sum of the three innermost* PD + PC + PB.

Put AP $= x$; then (*by Euc.* 47. 1.) BP $= \sqrt{100 + x^2}$ CP $= \sqrt{400 + x^2}$, &c. and, consequently, $\sqrt{100 + x^2} + \sqrt{400 + x^2} + \sqrt{900 + x^2} - \sqrt{1600 + x^2} -$

$\sqrt{2500 + x^2} = 0$. Now, by reflecting a little on the nature of the problem, it is easy to perceive that

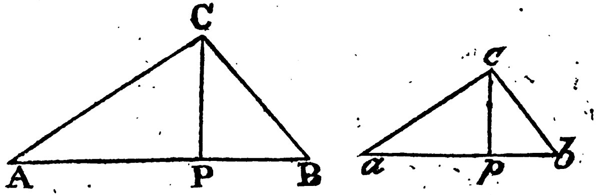


PF + PE must be greater than 90, seeing AF + AE is = 90; whence it appears that PD + PC + PB must also exceed 90, and that PC (considered as a mean between PD and PB) must be greater than 30; hence I conclude, that the value of AP, as it is something less than PC, will be somewhere about 30; and therefore I write $30 + e$ for x; and then, rejecting all the powers of e, above the first, as inconsiderable, our equation stands thus, $\sqrt{1000 + 60e} + \sqrt{1300 + 60e} + \sqrt{1800 + 60e} - \sqrt{2500 + 60e} - \sqrt{3400 + 60e} = 0$; which, by the method explained in page 174, will be transformed to $\sqrt{1000} + \frac{\sqrt{1000} \times 3e}{100} + \sqrt{1300} + \frac{\sqrt{1300} \times 3e}{130} + \sqrt{1800} + \frac{\sqrt{1800} \times 3e}{180} - 50 - 6e - \sqrt{3400} - \frac{\sqrt{3400} \times 3e}{340} = 0$; this, contracted, gives $1,8 + 1,37e = 0$; whence $e = -1,3$, and consequently $x = 28,7$, nearly. Let, now, 28,7 be put $= x$; and then, by proceeding as above, we shall have $,0083 + 1,43e = 0$; hence $e = -,0058$, and $x = 28,6942$; which is true to the last figure.

PROBLEM XXXII.

The perimeter AB + BC + AC, *and the perpendicular* CP, *of a triangle* ABC, *whose sides are in harmonic proportion* (AB : BC : : AB — AC : AC — BC) *being given; to determine the triangle.*

Let *abc* be another triangle, similar to the proposed one: and let $ab = 1$, $bc = x$, $ac = y$, $CP = a$, and $AB + BD + AC = b$: then, half the sum of the three sides of the triangle *abc* being $\frac{1 + x + y}{2}$, if from the same each particular side be subtracted, and all the remainders be multiplied continually together, and that product, again, by the said half sum, we shall have $\frac{1 + x - y}{2} \times \frac{1 + y - x}{2} \times \frac{y + x - 1}{2} \times \frac{1 + x + y}{2}$, equal to the second power of the area *abc* (*by prob.* 15): which, as the base is unity, also expresses $\frac{1}{4}$ of the square of the perpendicular. But the squares of the sides, as wellas the sides themselves of similar triangles, are proportional, and therefore $\overline{1 + x + y}]^2 : b^2 :: \frac{\overline{1 + x - y} \times \overline{1 - x + y} \times \overline{y + x - 1} \times \overline{1 + x + y}}{4} : a^2$; whence we have $4a^2 \times \overline{1 + x + y} = \overline{1 + x - y} \times \overline{1 - x + y} \times \overline{y + x - 1} \times bb$: but the sides AB, AC, and BC, being given in harmonic proportion, therefore, 1, y, and x, must likewise be in the same proportion; that is, $1 : x :: 1 - y : y - x$; whence $y - x = x - xy$, and therefore $y = \frac{2x}{1 + x}$; which, substituted above, gives $\frac{4a^2 \times \overline{1 + 4x + x^2}}{1 + x} = \frac{1 + x^2}{1 + x} \times \frac{1 + 2x - x^2}{1 + x} \times \frac{2x + x^2 - 1}{1 + x} \times b^2$, or $4a^2 \times \overline{1 + 4x + x^2} \times \overline{1 + x}]^2 = \overline{1 + x^2} \times \overline{1 + 2x - x^2} \times \overline{2x + x^2 - 1} \times b^2$; from which x will be found, and

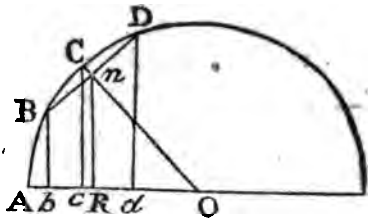


also y $(= \frac{2x}{1+x})$; and from thence the required sides of the similar figure ABC will, by proportion, be likewise known.

PROBLEM XXXIII.

Let there be three equi-different arches, AB, AC, *and* AD; *and, supposing the sine and co-sine of the mean* AC, *of the lesser extreme* AB, *and of the common difference* BC (*or* CD) *to be given, it is proposed to find the sine and co-sine of the greater extreme* AD.

Upon the radius AO let fall the perpendiculars Bb, Cc, and Dd; join B, D, and from the center O let the radius OC be drawn, cutting BD in n; also draw nR parallel to Cc, meeting AO in R: then, because of the similar triangles OCc and OnR, it will be, OC : On : : Cc : nR; and OC : On : : Oc : OR: whence we have $n\text{R} = \frac{\text{C}c \times \text{O}n}{\text{OC}}$, and $\text{OR} = \frac{\text{O}c \times \text{O}n}{\text{OC}}$: but, since BC is equal to CD (and therefore Bn equal to Dn), nR will, it is plain, be an arithmetical mean between Bb and Dd, and so is equal to half their sum, or $\frac{\text{B}b + \text{D}d}{2}$: and, for the very same reason, OR will be equal to $\frac{\text{O}b + \text{O}d}{2}$; consequently, $\frac{\text{B}b + \text{D}d}{2} = \frac{\text{C}c \times \text{O}n}{\text{OC}}$, and $\frac{\text{O}b + \text{O}d}{2} = \frac{\text{O}c \times \text{O}n}{\text{OC}}$; whence $\text{D}d = \frac{\text{C}c \times 2\text{O}n}{\text{OC}} - \text{B}b$, and $\text{OD} = \frac{\text{O}c \times 2\text{O}n}{\text{OC}} - \text{O}b$; which, if the radius OC be supposed unity, will become $\text{D}d = \text{C}c \times 2\text{O}n - \text{B}b$; and $\text{O}d = \text{O}c \times 2\text{O}n - \text{O}b$: from whence we have the two following theorems.



Theor. 1. *If the sine of the mean of any three equi-different arches (the radius being supposed unity), be multiplied by twice the co-sine of the common difference, and from the product the sine of either extreme be subtracted, the remainder will be the sine of the other extreme.*

Theor. 2. *And if the co-sine of the mean of three equi-different arches be multiplied by twice the co-sine of the common difference, and the co-sine of either extreme be subtracted from the product, the remainder will be the co-sine of the other extreme.*

PROBLEM XXXIV.

The sine and co-sine of an arch being given, to find the sine and co-sine of any multiple of that arch.

Let the given arch be represented by A, its sine by x, and co-sine by $\frac{1}{2}y$, the radius being unity. Then, since the arch A may be considered as an arithmetical mean between 0 and 2A, we shall, by the first of the two preceding theorems, have

sine of 2A (= sine of A $\times\, y$ — sine of 0) $= xy$;
sine of 3A (= sine of 2A $\times\, y$ — sine of A) $= xy^2 - x$;
sine of 4A (= sine of 3A $\times\, y$ — sine of 2A $= xy^3 - xy - xy) = xy^3 - 2xy$;
sine of 5A (= sine of 4A $\times\, y$ — sine of 3A $= xy^4 - 2xy^2 - xy^2 + x) = xy^4 - 3xy^2 + x$;
sine of 6A (= sine of 5A $\times\, y$ — sine of 4A $= xy^5 - 3xy^3 + xy - xy^3 + 2xy) = xy^5 - 4xy^3 + 3xy$;
sine of 7A (= sine of 6A $\times\, y$ — sine of 5A $= xy^6 - 4xy^4 + 3xy^2 - xy^4 + 3xy^2 - x) = xy^6 - 5xy^4 + 6xy^2 - x$:

whence, *universally*, the sine of the multiple arch nA, where n denotes any whole positive number whatever, will be truly expressed by $x\ \times$ into this series:

$$y^{n-1} - \frac{n-2}{1} \times y^{n-3} + \frac{n-3}{1} \times \frac{n-4}{2} \times y^{n-5} - \frac{n-4}{1} \times \frac{n-5}{2} \times \frac{n-6}{3} \times y^{n-7}$$

&c. Moreover, from the second theorem, we have co-sine of 2A (= co-sine of A $\times\, y$ — co-sine of 0 $= \frac{y^2}{2} - 1) = \frac{y^2 - 2}{2}$;

Co-sine of 3A $(=$ co-sine of 2A $\times y -$ co-sine of A $= \frac{y^3}{2} - y - \frac{y}{2}) = \frac{y^3 - 3y}{2}$;

Co-sine of 4*a* $(=$ co-sine of 3A $\times y -$ co-sine of 2A $= \frac{y^4 - 3y^2}{2} - \frac{y^2 - 2}{2}) = \frac{y^4 - 4y^2 + 2}{2}$;

whence, *universally*, the co-sine of the multiple arch *n*A will be truly represented by $\frac{y^n}{2} - \frac{ny^{n-2}}{2} + \frac{n}{2} \times \frac{n-3}{2} \times y^{n-4} - \frac{n}{2} \times \frac{n-4}{2} \times \frac{n-5}{3} \times y^{n-6} + \frac{n}{2} \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} \times y^{n-8}$, &c. which series, as well as that for the sine, is to be continued till the indices of y become nothing, or negative.

But, if you would have the sine expressed in terms of x *only*, then, because the square of the sine + the square of the co-sine is always equal to the square of the radius, and, therefore, in this case, $x^2 + \frac{1}{4}y^2 = 1$, it is manifest that the sines of all the odd multiples of the given arch A, wherein only the even powers of y enter, may be exhibited in terms of x only, without surd quantities: so that $4 - 4x^2$ being substituted for its equal y^2, in the sines of the aforementioned arches, we shall have

1st. Sine of 3A $= 3x - 4x^3$;
2d. Sine of 5A $= 5x - 20x^3 + 16x^5$;
3d. Sine of 7A $= 7x - 56x^3 + 112x^5 - 64x^7$;
4th. Sine of 9A $= 9x - 120x^3 + 432x^5 - 576x^7 + 256x^9$;
&c. &c.

And, *generally*, if the multiple arch be denoted by *n*A, then the sine thereof will be truly represented by $nx - \frac{n}{1} \times \frac{n^2-1}{2\,.\,3} \times x^3 + \frac{n}{1} \times \frac{n^2-1}{2\,.\,3} \times \frac{n^2-9}{4\,.\,5} \times x^5 - \frac{n}{1} \times \frac{n^2-1}{2\,.\,3} \times \frac{n^2-9}{3\,.\,4} \times \frac{n^2-25}{5\,.\,6} \times x^7 + \frac{n}{1} \times \frac{n^2-1}{2\,.\,3} \times \frac{n^2-9}{4\,.\,5} \times \frac{n^2-25}{6\,.\,7} \times \frac{n^2-49}{8\,.\,9} \times x^9$, &c.

From this series, the sine of the sub-multiple of any arch, where the number of parts is odd, may also be found, supposing (s) the sine of the whole arch to be given: for let x be the required sine of the sub-multiple, and n the number of equal parts into which the whole arch is divided; then, by what has been already shown, we shall have $nx - \frac{n}{1} \times \frac{n^2-1}{2.3} \times x^3 + \frac{n}{1} \times \frac{n^2-1}{2.3} \times \frac{n^2-9}{4.5} \times x^5$, &c. $= s$: from the solution of which equation the value of x will be known. Hence, also, we have an equation for finding the side of a regular polygon inscribed in a circle: for, seeing the sine of any arch is equal to half the chord of double that arch, let $\frac{1}{2}v$ and $\frac{1}{2}w$ be written above for x and s, respectively, and then our equation will become $\frac{nv}{2} - \frac{n}{1} \times \frac{n^2-1}{2.3} \times \frac{v^3}{8} + \frac{n}{1} \times \frac{n^2-1}{2.3} \times \frac{9^2-9}{4.5} \times \frac{v^5}{32}$, &c. $= \frac{1}{2}w$, or $nv - \frac{n}{1} \times \frac{n^2-1}{2.3} \times \frac{v^3}{4} + \frac{n}{1} \times \frac{n^2-1}{2.3} \times \frac{n^2-9}{4.5} \times \frac{v^5}{16}$, &c. $= w$; expressing the relation of chords, whose corresponding arches are in the ratio of 1 to n. But, when the greater of the two arches becomes equal to the whole periphery, its chord (w) will be nothing, and then the equation, by dividing the whole by nv, will be reduced to $1 - \frac{n^2-1}{2.3} \times \frac{v^2}{4} + \frac{n^2-1}{2.3} \times \frac{n^2-9}{4.5} \times \frac{v^4}{16} - \frac{n^2-1}{2.3} \times \frac{n^2-9}{4.5} \times \frac{n^2-25}{5.6} \times \frac{v^6}{64}$, &c. $= 0$; where n is the number of sides, and v the side of the polygon.

From the foregoing series, that given by sir *Isaac Newton*, in *Phil. Trans.* mentioned in p. 242 of this Treatise, may also be easily derived. For, if the arch A and its sine x be taken indefinitely small, they will be to one another in the ratio of equality, indefinitely near, by what has been proved at p. 246; in which case, the general expression, by writing A instead of x, will become $nA - \frac{n}{1} \times \frac{n^2 - 1}{2 \cdot 3} \times A^3 + \frac{n}{1} \times \frac{n^2 - 1}{2 \cdot 3} \times \frac{n^2 - 9}{4 \cdot 5}$ $A^5 - \frac{n}{1} \times \frac{n^2 - 1}{2 \cdot 3} \times \frac{n^2 - 9}{4 \cdot 5} \times \frac{n^2 - 25}{6 \cdot 7} \times A^7$, &c.

Therefore, if n be now supposed indefinitely great, so that the multiple arch nA may be equal to any given arch z, the squares of the odd numbers, 1, 3, 5, &c. in the factors $n^2 - 1$, $n^2 - 9$, $n^2 - 25$, &c. may be rejected as nothing, or inconsiderable, in respect of n^2; and then the foregoing series will become $nA - \frac{n^3A^3}{2 \cdot 3} + \frac{n^5A^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{n^7A^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$, &c. wherein, if for nA, its equal, z, be substituted, we shall then have $z - \frac{z^3}{2 \cdot 3} + \frac{z^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{z^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$, &c. which is the sine of the arch z, and the same with that before given.

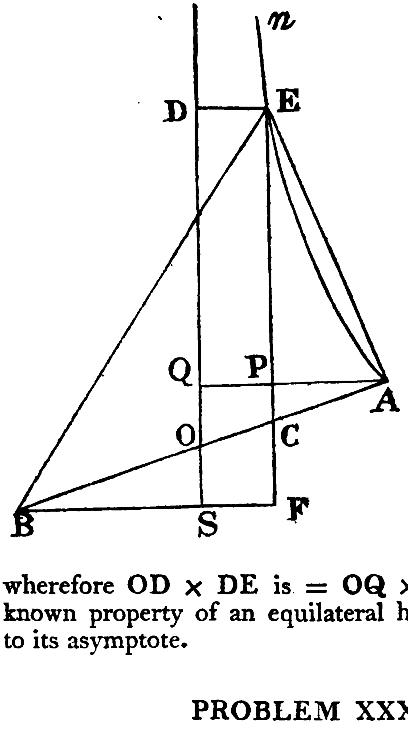
Moreover, the aforegoing general expressions may be applied, with advantage, in the solution of cubic, and certain other higher equations, included in this form, *viz.* $z^n - az^{n-2} + \frac{n-3}{2n} \times a^2z^{n-4} - \frac{n-4}{2n} \times \frac{n-5}{3n} \times a^3z^{n-6} + \frac{n-5}{2n} \times \frac{n-6}{3n} \times \frac{n-7}{4n} \times a^4z^{n-8}$, &c. $= f$. For, if z be put $= y\sqrt{\frac{a}{n}}$, the equation will be transformed to $\left.\overline{\frac{a}{n}}\right|^{\frac{n}{}} \times$ into this series $y^n - ny^{n-2} + \frac{n}{1} \times \frac{n-3}{2} \times y^{n-4} + \frac{n}{1} \times \frac{n-4}{2} \times \frac{n-5}{3} \times y^{n-6}$ &c. $= f$, and consequently $\frac{y^n}{2} - \frac{n}{2} \times y^{n-2} + \frac{n}{2} \times \frac{n-3}{2}$

$\times\, y^{n-4} - \frac{n}{2} \times \frac{n-4}{2} \times \frac{n-5}{3} \times y^{n-6}$, &c. $= \frac{\frac{1}{2}f}{\left.\frac{a}{n}\right|^{\frac{n}{2}}}$: from whence, as it is proved above, that the former part of the equation (and therefore its equal) represents the co-sine of n times the arch whose co-sine is $\frac{1}{2}y$, we have the following rule:

Find, from the tables, the arch whose natural co-sine is $\frac{\frac{1}{2}f}{\left.\frac{a}{n}\right|^{\frac{n}{2}}}$ *(or its log. co-sine* $=$ *log.* $\frac{1}{2}f - \frac{n}{2}$ *log.* $\frac{a}{n}$*) the radius being unity; take the* nth *part of that arch, and find its co-sine, which multiply by* $2\sqrt{\frac{a}{n}}$*, and the product will be the true value of* z*, in the proposed equation* $z^n - az^{n-2} + \frac{n-3}{2n} \times a^2z^{n-4} - \frac{n-4}{2n} \times \frac{n-5}{3n} \times a^3z^{n-6}$*, &c.*

Thus, let it be required to find the value of z, in the cubic equation $z^3 - 432z = 1728$; then, we shall have $n = 3$, $a = 432$, and $f = 1728$; consequently $\frac{\frac{1}{2}f}{\left.\frac{a}{n}\right|^{\frac{n}{2}}}$ $\left(= \frac{864}{\left.144\right|^{\frac{3}{2}}}\right) = ,5$, and the arch corresponding thereto $= 60°$; whence the co-sine of (20°) $\frac{1}{3}$ thereof will be found ,9396926; and this, multiplied by 24 $\left(= 2\sqrt{\frac{a}{n}}\right)$, gives 22,55262 for one value of z. But besides this, the equation has two other roots, both of which may be found after the very same manner: For, since 0,5 is not only the co-sine of 60°, but also of $60° + 360°$, and $60° + 2 \times 360°$, let the co-sine of (140°) $\frac{1}{3}$ of the former of these arches be now taken; which is $-$,7660444, and must be expressed with a negative sign, because the arch corresponding is greater than one right angle, and less than three. Then, the value

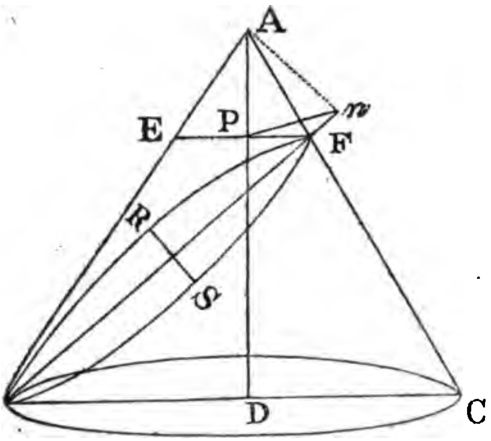
ACE, as CAE exceeds CBE, it is evident that the sum of the two angles BCE, CBE, of the triangle BCE, is equal to the sum of the two angles ACE, CAE of the triangle ACE; and consequently, that the remaining angles AEC and BEC, are equal the one to the other: therefore, by reason of the similar triangles EFB, EAP, we have $EF : EP :: BF : AP$, that is, $OD + OQ : OD - OQ :: QA + DE : QA - DE$; whence, by composition and division, $2OD : 2OQ :: 2QA : 2DE$; wherefore $OD \times DE$ is $= OQ \times QA$; which is the known property of an equilateral hyberbola with respect to its asymptote.



PROBLEM XXXVII.

To find the solidity of a conical ungula BFCB, *cut off by a plane* BRFSB *passing through one extremity of the base diameter.*

Let EPF be parallel to the base diameter BC, cutting AD the axis of the cone in P; also, let An be perpendicular to BF; join P, n, and let RS be the conjugate axis of the elliptical section BRFSB: then the part ABF, above the said section, being an oblique elliptical cone, its solidity will be expressed by $,7854 \times SR \times BF \times \frac{An}{3}$, that is, by the area of its base BRFSB drawn into one-third of the perpendicular height. But the triangles BCF and AP*n* will appear to be equi-angular; for, APF and A*n*F being both right angles, the circumference of a circle, described on the diameter AF, will pass through P and *n*; and so the angles AF*n* (BFC) and AP*n*, as well as AFP (FCB) and A*n*P,



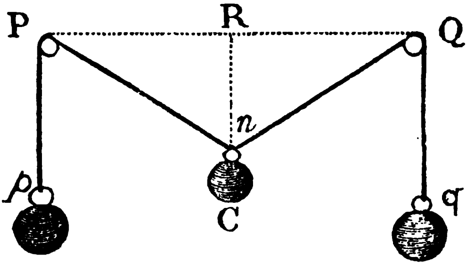
insisting on the same arch, are, respectively, equal. Hence we have BC : BF : : A*n* : AP; and therefore BF × A*n* = BC × AP: this value being substituted above, the content of the part ABF becomes SR × BC × AP × ,2618: which, because SR is known to be $= \sqrt{BC \times EF}$, is farther reduced to $BC \times AP \times \sqrt{BC \times EF} \times ,2618$. This, subtracted from $BC^2 \times AD \times ,2618$, the content of the whole cone ABC, leaves

$\overline{BC^2 \times AD - BC \times AP \times \sqrt{BC \times EF}} \times ,2618$ for the required solidity of the *ungula* BCF; which, because $AD = \frac{DP \times BC}{BC - EF}$, and $AP = \frac{DP \times EF}{BC - EF}$, will be reduced to $\frac{BC^2 - EF \times \sqrt{BC \times EF}}{BC - EF} \times ,2618\ DP \times BC$.

PROBLEM XXXVIII.

Let A *and* B *be two equal weights, made fast to the ends of a thread, or perfectly flexible line* $pPnQq$, *supported by two pins, or tacks,* P, Q, *in the same horizontal plane; over which pins the line can freely slide either way; and let* C *be another weight, fastened to the thread, in the middle, between* P *and* Q: *now the question is, to find the position of the weight* C, *or its distance below the horizontal line* PQ, *to retain the other two weights* A *and* B *in equilibrio.*

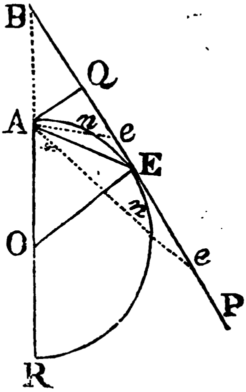
Let PR ($= \frac{1}{2}$PQ) be denoted by a, and Rn (the distance sought) by x; and then Pn, or Qn, will be represented by $\sqrt{a^2 + x^2}$. Therefore, *by the resolution of forces*, it will be, as $\sqrt{a^2 + x^2}$ (Pn) : x (Rn) : : the whole force of the weight A in the direction Pn, : $\frac{Ax}{\sqrt{a^2 + x^2}}$, its force in the direction nR, whereby it endeavours to raise the weight C; which quantity also expresses the force of the weight B in the same direction: but the sum of these two forces, since the weights are supposed to rest in equilibrio, must be equal to that of the weight C; that is, $\frac{2Ax}{\sqrt{a^2 + x^2}} = C$; whence we have $4A^2x^2 = C^2a^2 + C^2x^2$, and consequently $x = \frac{aC}{\sqrt{4A^2 - C^2}}$.



PROBLEM XXXIX.

To determine the position of an inclined plane AE, *along which a heavy body, descending by the force of its own gravity from a given point* A, *shall reach a right line* BP, *given by position, in the least time possible.*

Through the given point A, perpendicular to the horizon, let there be drawn the right line RB, meeting BP in B; also conceive the semicircle AER to be described, touching BP in E; then let AE be drawn, which will be the position required; because the time of descent along the chord AE being equal to that along any other chord A*n*, it will consequently be less than the time of the descent along A*e*, whereof A*n* is only a part: therefore, if AQ and OE be now made perpendicular to BP, we shall have (by reason of the similar triangles) $AB : AQ :: AB + AO : (OE)\ AO$; whence, by multiplying extremes and means, $AB \times AO = AQ \times AB + AQ \times AO$; therefore $AB \times AO - AQ \times AO = AQ \times AB$, and $AO\ (OE) = \frac{AQ \times AB}{AB - AQ}$; from which BE and AE are also given.

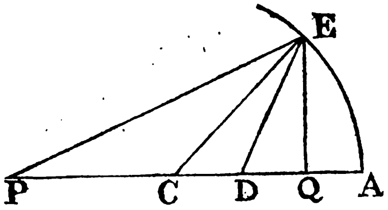


The geometrical construction of this problem is extremely easy; for, if AQ (as above) be drawn perpendicular to BP, and the angle OAQ be bisected by AE, the thing is done: because, OE being drawn parallel to AQ, the angle OEA is = QAE = EAO; and so, AO being = OE, the semicircle that touches BP will pass through A.

PROBLEM XL.

A ray of light, from a lucid point P, *in the axis* AP, *of a concave spherical surface, is reflected at a given point* E *in that surface; to find the point* D *where the reflected ray meets the axis.*

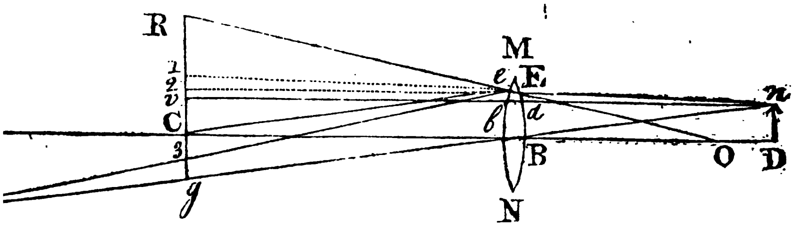
Draw EQ perpendicular to AP, and from the center C let CE be drawn; also make CE $= a$, CQ $= b$, CP $= c$, CD $= x$; and (*by Euc.* 12. 3.) PE will be $= \sqrt{a^2 + c^2 + 2bc}$: wherefore, the angles of incidence and reflection CEP and CED being equal, we have, as PC $(c$: CD (x) : : PE $(\sqrt{a^2 + c^2 + 2bc})$: ED $= \frac{x\sqrt{a^2 + c^2 + 2bc}}{c}$; also, for the same reason, we have $\overline{PE \times ED} - PC \times CD = EC^2$, that is, $\frac{x \times \overline{a^2 + c^2 + 2bc}}{c} - cx = a^2$; which, reduced, gives $x = \frac{ca^2}{a^2 + 2bc}$, showing how far from the center the ray cuts the axis. But if the lucid point P be supposed infinitely remote, so that the ray PE may be considered as parallel to the axis AP, the expression will be more simple; for then a^2, in the divisor, may be rejected as nothing in comparison of $2bc$; that being done, CD, or x, becomes $= \frac{a^2}{2b}$; which, therefore, if E be taken near the vertex A, will be $= \frac{1}{2}a$, very nearly.



PROBLEM XLI.

To find the magnitude and position of an image formed by refraction at a given lens.

Let MN be the given lens, DOBCF the axis thereof, and D*n* the object whose image FH we would find; also let CB be the radius of that surface of the lens MBN, which is nearest the object, and OB that of the other surface: make RCg perpendicular to DF, and from



n, to any point E in the surface of the lens, draw the incident ray *n*E, and let the continuation thereof be E1, and let the direction of the same ray, after the first refraction at E, be E2; and, after it is refracted a second time, at *e*, let its direction be *e*3H; draw CE and O*e*R, and make *ndv* parallel to DF, calling O*b*, *b*; CB, *c*; BD, *d*; D*n*, *p*; and the distance of the point E from the axis DF, *x*; and let the sine of incidence be to the sine of refraction, out of air into glass, as *m* to *n*. Then, the thickness of the lens being looked upon as inconsiderable in respect of the focal distance F*b*, we shall have, as $d : x - p$ (E*d*) : : $d + c$ (*nv*) : $\frac{dx + cx - dp - cp}{d} = v1$; which added to C*v* (*p*) gives $\frac{dx + cx - cp}{d} = C1$: therefore $m : n :: \frac{dx + cx - cp}{d} : \frac{n \times \overline{dx + cx - cp}}{dm} = C2$. Moreover, *b* (O*b*) : *x* (BE) : : $b + c$ (OC) : $\frac{\overline{b + c} \times x}{b} = CR$; whence R2 (= CR − C2) $= \frac{\overline{b + c} \times x}{b} - n \times \frac{dx + cx - cp}{dm}$: but $n : m ::$ R2 : R3 $= \frac{mx \times \overline{b + c}}{bn} + \frac{cp - cx - dx}{d}$; and therefore

$$C3\ (=R3-RC)=\frac{\overline{m-n}\times x\times\overline{b+c}}{bn}+\frac{cp-cx-dx}{d}.$$
Let x be now taken $=0$, and then C3 will become $\frac{cp}{d}$, which let be represented by C3, and draw Bg, producing the same till it meets $e3$, produced, in H: then $g3$ being $(=Cg-C3)=\frac{\overline{d+c}\times x}{d}-\frac{\overline{m-n}\times\overline{b+c}\times x}{bn}$, and the triangles H3$g$ and HEB equiangular, it will be, as $(EB-g3)\ \frac{\overline{m-n}\times\overline{b+c}\times x}{bn}-\frac{cx}{d}:(Bg)\ c::(EB)\ x:BH=\frac{nbcd}{\overline{m-n}\times\overline{b+c}\times d-nbc}=$ the required distance of the image from the lens; and as $c\ (BC):\frac{cp}{d}\ (CG)::BH\ (\text{or } BF):FH=\frac{BF\times p}{BD}=\frac{nbcp}{\overline{m-n}\times\overline{b+c}\times d-nbc}$ (= HF) the magnitude or the image, or its linear amplification.

COROL. 1.

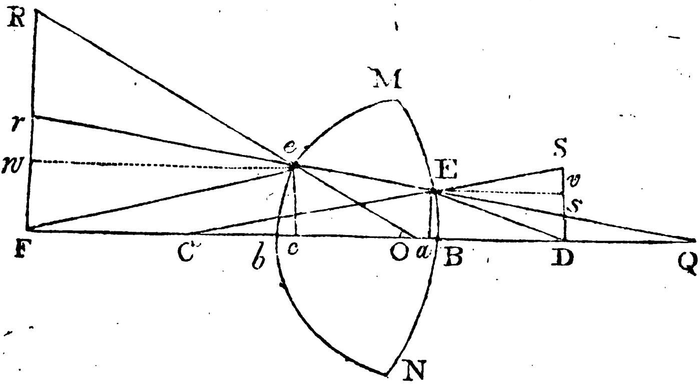
Because the values of BH and HF are alike affected by b and c, it follows that both the distance and magnitude of the image will remain unaltered, if the place of the lens be the same, let which side you will be turned towards the object.

COROL. 2.

If d be made infinite, or the distance of the object from the lens be supposed indefinitely great, BF will become $\frac{nbc}{\overline{m-n}\times\overline{b+c}}$; which is the principal focal distance, at which the parallel rays unite, and this distance, when both sides of the lens have the same convexity, or b is $=c$, will become $=\frac{nb}{2m-2n}$: but in a *plano-convex*, where b is infinite, it will be $=\frac{nc}{m-n}$, and, in a *meniscus*, where b is negative, or one surface concave, and the other convex, it will be $\frac{nbc}{\overline{m-n}\times\overline{b-c}}$.

The same answered otherwise, allowing also for the thickness of the lens.

Supposing, as before, that F is the place of the image of an object at D, let FR and DS be supposed perpendicular to the axis FBQ, intersecting the continuation of Ee (the intercepted part of the ray DEeF) in r and s, and meeting the radii Oe, CE (produced) in R and S; likewise let Ea and ec be perpendicular to QF, and Ev and Ew parallel thereto: then, because the ray is supposed to be indefinitely near the axis, ac may be taken for the thickness of the lens, which let be denoted by t; putting bF $= z$, $ce = y$, aE $= x$, O$b = b$, CB $= c$, and BD $= d$ (*as before*). By similar triangles, Ca (c) : aE (x) : : CD ($c + d$) : DS $= \frac{x\times\overline{c+d}}{c}$; and, by the law of refraction, $m : n :: $ DS : Ss $= \frac{n}{m}\times\frac{x\times\overline{c+d}}{c}$; whence D$s$ (= DS − Ss) $= \overline{1-\frac{n}{m}}\times\frac{x\times\overline{c+d}}{c}$, and vs (= aE − Ds) =



$x \times \overline{r - \frac{qd}{c}}$, by making $r = \frac{n}{m}$, and $q = 1 - r$.

Now, vs : Ev (BD) : : aE : aQ (BQ); that is, $x \times \overline{r - \frac{qd}{c}} : d :: x : \frac{cd}{er - qd} =$ BQ; which is given from hence.

Again, in the very same manner, Oc (b) : ce (y) : : OF $(b + z)$: FR $= \frac{y \times \overline{b + x}}{b}$; and $m : n ::$ FR : R$r = \frac{n}{m} \times \frac{y \times \overline{b + z}}{b}$: whence F$r = \overline{1 - \frac{n}{m}} \times \frac{y \times \overline{b + z}}{b} = \frac{qy \times \overline{b + z}}{b}$, and wr (F$r - ce$) $= y \times \overline{\frac{qz}{b} - r}$; and therefore cQ $\left(= \frac{we \times ce}{wr}\right) = \frac{bz}{qz - br}$: from which subtracting the value of BQ, found above, we get this equation, *viz.* $\frac{bz}{qz - br} - \frac{cd}{cq - qd} = t$: whence the value of z, by making the given quantity $t + \frac{cd}{rc - qd} = g$, comes out $= \frac{rbg}{qg - b}$. But, if you had rather have the same in original terms, it is but substituting for g; whence, after reduction, $z = \frac{rbcd + rbt \times \overline{rc - qd}}{qd \times \overline{b + c} - rbc + qt \times \overline{rc - qd}}$: which, by restoring m and n, becomes

$$z = \frac{mnbcd + nbt \times \overline{nc - \overline{m - n} \times d}}{\overline{m - n} \times md \times \overline{b + c} - mnbc + \overline{m - n} \times t \times \overline{nc - \overline{m - n} \times d}};$$

where, if t be taken $= 0$, we shall have

$z = \frac{nbcd}{\overline{m - n} \times d \times \overline{b + c} - nbc}$, the very same as found by the preceding method.

APPENDIX:

CONTAINING THE

CONSTRUCTION

OF

GEOMETRICAL PROBLEMS.

WITH

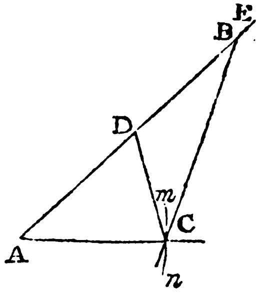
THE MANNER OF RESOLVING THE SAME
NUMERICALLY.

PROBLEM I.

The base, the sum of the two sides, and the angle at the vertex of any plane triangle being given; to describe the triangle.

CONSTRUCTION.

DRAW the indefinite right line AE, in which take AB equal to the sum of the sides, and make the angle ABC equal to half the given angle at the vertex, and upon the point A, as a center, with a radius equal to the given base, let a circle nCm be described, cutting BC in C; join A, C, and make the angle BCD = CBD, and let CD cut AB in D; then will ACD be the triangle that was to be constructed.



DEMONSTRATION.

Because the angles BCD and CBD are equal, therefore is CD = DB (*Euc.* 6. 1.), and consequently AD + DC = AB: likewise, for the same reason, the angle ADC = BCD + CBD (*Euc.* 32. 1.) is equal to 2CBD. *Q. E. D.*

Method of Calculation.

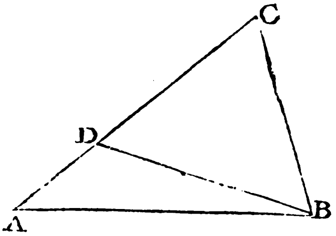
In the triangle ABC are given the two sides AB, AC, and the angle ABC, whence the angle A is known; then in the triangle ADC will be given all the angles, and the base AC; whence the sides AD and DC will also be known.

PROBLEM II.

The angle at the vertex, the base, and the difference of the sides, being given; to determine the triangle.

CONSTRUCTION.

Draw AC at pleasure, in which take AD equal to the difference of the sides, and make the angle CDB equal to the complement of half the given angle to a right angle; then from the point A draw AB equal to the given base, so as to meet DB in B, and make the angle DBC = CDB, then will ABC be the triangle required.



DEMONSTRATION.

Since (*by construction*) the angles CDB and DBC are equal, CB is equal to CD, and therefore CA — CB = AD: moreover, each of those equal angles being equal to the complement of half the given angle, their sum, which is the supplement of the angle C, must therefore be equal to two right angles, — the (whole) given angle, and consequently C = the given angle. *Q. E. D.*

Method of Calculation.

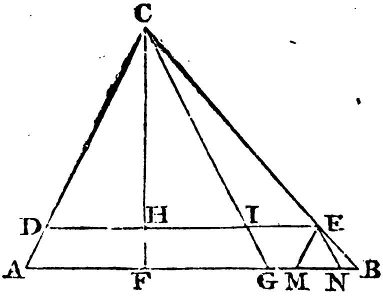
In the triangle ABD are given the sides AB, AD, and the angle ADB; whence the angle A will be given, and consequently BC and AC.

PROBLEM III.

The angle at the vertex, the ratio of the including sides, and either the base, the perpendicular, or difference of the segments of the base, being given; to describe the triangle.

CONSTRUCTION.

Draw CA at pleasure, and make the angle ACB equal to the angle given; take CB to CA in the given ratio of the sides, and join A, B: then, if the base be given, let AM be taken equal thereto, and draw ME parallel to CA meeting CB in E, and make ED parallel to AB; but, if the perpendicular be given, let fall CF perpendicular to AB, in which take CH equal to the given perpendicular, and draw DHE parallel to AB; lastly, if the difference of the segments of the base be given, take FG = AF, and join C, G, and take GN equal to the difference of the segments given, drawing NE parallel to CG, and ED to BA (as before); then will CDE be the triangle which was to be constructed.



DEMONSTRATION.

Because of the parallel lines AB, DE; ME, AC; and NE, GC; thence is DE = AM, and EI = NG; and also CD : CE : : CA : CB (*Euc.* 4. 6.). *Q. E. D.*

Method of Calculation.

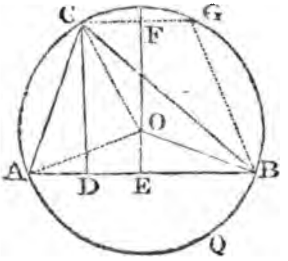
Let AC be assumed at pleasure; then the ratio of AC to BC being given, BC will become known; and therefore in the triangle ACB will be given two sides and the included angle; whence the angles B and A, or E and D, will be found; then in the triangle EDC, EHC, or EIC, according as the base, perpendicular, or the difference of the segments of the base is given, you will have one side and all the angles; whence the other sides will be known.

PROBLEM IV.

The angle at the vertex, and the segments of the base, made by a perpendicular falling from the said angle, being given; to describe the triangle.

CONSTRUCTION.

Let the given segments of the base be AD and DB; bisect AB by the perpendicular EF, and make the angle EBO equal to the difference between the given angle and a right one, and let BO meet EF in O; from O, as a center, with the radius OB, describe the circle BGAQ, and draw DC perpendicular to AB, meeting the periphery of the circle in C; join A, C, and C, B, then will ACB be the triangle that was to be constructed.



DEMONSTRATION.

The angle ACB, at the periphery, standing upon the arch AQB, is equal to EOB, half the angle at the center, standing upon the same arch; but EBO is equal to the difference of the given angle and a right one (*by construction*); therefore ACB (EOB) is equal to the angle given. *Q. E. D.*

Method of Calculation.

Draw CFG parallel to AB; then it will be, as the base AB : the difference of segments CG (: : EB : CF) : : the sine of the given angle at the vertex (EOB) :

the sine of (COF = CBG) the difference of the angles at the base; whence the angles themselves are given.

After the same manner, a segment of a circle may be described to contain a given angle, when that angle is greater than a right one, if, instead of BO being drawn above AB, it be taken on the contrary side.

PROBLEM V.

Having given the base, the perpendicular, and the angle at the vertex of any plane triangle, to construct the triangle.

CONSTRUCTION.

Upon AB, the given base (*see the preceding figure*), let the segment ACGB of a circle be described to contain the given angle, as in the last problem; take EF equal to the given perpendicular, and draw FC parallel to AB, cutting the periphery of the circle in C; join A, C, and B, C, and the thing is done; the demonstration whereof is evident from the last problem.

Method of Calculation.

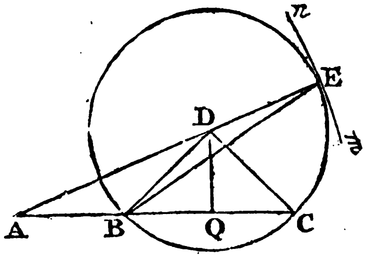
In the triangle EBO are given all the angles and the side EB; whence EO will be known, and consequently OF (= DC — EO); then it will be as EB : OF : : the sine of EOB (the given angle at the vertex) : the sine of OCF, the complement of (COF or CBG) the difference of the angles at the base; whence these angles themselves are likewise given. This calculation is adapted to the logarithmic canon; but, by means of a table of natural sines, the same result may be brought out by one proportion only: for BE being the sine of BOE, and OE and OF co-sines of BOE and COF (answering to the equal radii OB and OC), it will therefore be, BE : EF : : *sine* BOE (ACB) : *co-sine* BOE + *co-sine* COF; from which, by subtracting the co-sine of BOE, the co-sine of COF (= CBG) is found.

PROBLEM VI.

The angle at the vertex, the sum of the two including sides, and the difference of the segments of the base being given, to describe the triangle.

CONSTRUCTION.

Draw the right line AC at pleasure, in which take AB equal to the difference of the segments of the base, and make the angle CBE equal to half the supplement of the given angle; and from A to BE apply AE equal to the given sum of the sides; make the angle EBD = BED, and let BD meet AE in D, and from the center D, with the radius DB, describe the circle DBC, cutting AC in C, and join D, C; then will ACD be the triangle required.



DEMONSTRATION.

The angle EBD being = BED, therefore is DE = DB = DC, and consequently AD + DC = AE. Moreover, the angle CDE, at the center, is double to the angle CBE, at the periphery, both standing upon the same arch CE; which last (*by construction*) is equal to half the supplement of the given angle; therefore CDE is equal to the whole supplement, and consequently ADC equal to the given angle itself. *Q. E. D.*

Method of Calculation.

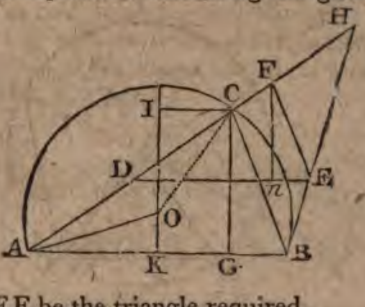
In the triangle ABE are given the two sides AB, AE, and the angle ABE; whence the angle A will be given; then in the triangle ABD will be given all the angles and the side AB; whence AD and DC (DB) will be also given.

PROBLEM VII.

The angle at the vertex, the sum of the including sides, and the ratio of the segments of the base being given; to determine the triangle.

CONSTRUCTION.

Let AG be to GB in the given ratio of the segments of the base, and, upon the right line AB, let a segment of a circle be described, capable of containing the given angle; draw GC perpendicular to AB, meeting the periphery in C; join A, C, and C, B, and in AC, produced, take CH = CB; join B, H, and in HA take HD equal to the given sum of the sides, draw DE parallel to AB, and EF to BC; then will DEF be the triangle required.



DEMONSTRATION.

Let Fn be perpendicular to DE. Whereas (*by construction*) CH is equal to CB, and FE parallel to CB, therefore is FE = FH (*Euc.* 4. 6.) and consequently FE + FD = HD: also, because FE is parallel to CB, therefore is the angle DFE = ACB: moreover, the triangles ABC, DEF, being equiangular, it will be, as AG : GB : : Dn : nE. *Q. E. D.*

Method of Calculation.

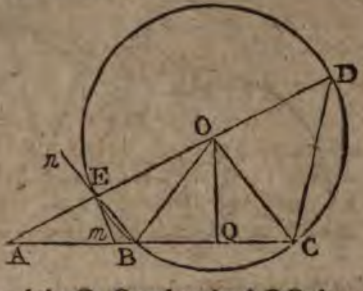
From the center O, conceive AO and OC to be drawn; supposing KOI perpendicular, and CI parallel to AB: then it will be, as AK is to CI (KG) so is the sine of AOK (= ACB, *see prob.* 4.) to the sine of COI, the difference of the angles ABC and BAC; which are both given from hence, because their sum is given by the question: therefore in the triangle DHE are given all the angles and the side HD, whence the base DE will be known.

PROBLEM VIII.

Having the angle at the vertex, the difference of the including sides, and the difference of the segments of the base, to describe the triangle.

CONSTRUCTION.

Take AB equal to the difference of the segments of the base, and make the angle ABn equal to half the given angle; from A to B*n* apply AE = the difference of the sides; produce AE, and make the angle EBO = BEO, and let BO meet AE, produced in O, and from the center O, at the distance of OB, describe the circumference of a circle, cutting AB produced in C; join O, C; then is AOC the triangle sought.



DEMONSTRATION.

Because the angle EBO is = BEO (*by construction*); therefore is EO = BO = CO, and consequently AO — OC = AE. Furthermore, because the angle AOC is double to ADC, and ADC = ABE (*Euc. corol.* 22. 3.), therefore is AOC also double to ABE. *Q. E. D.*

Method of Calculation.

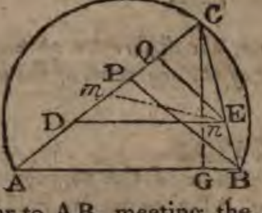
The two sides AB, AE, and the angle ABE being given, the angle A will from thence be found; then in the triangle ABO will be given all the angles and the side AB, whence OB (OC) and OA will be known.

PROBLEM IX.

The angle at the vertex, the difference of the including sides, and the ratio of the segments of the base, being given, to determine the triangle.

CONSTRUCTION.

Let AG be to GB in the given ratio of the segments of the base, and upon the right line AB let a segment of a circle ACB be described (*by prob.* 4.) capable of the given angle; draw GC perpendicular to AB, meeting the periphery in C, and join A, C,



and B, C; in AC take AP = BC, and draw BP; also, in AC, take CQ equal to the given difference of the sides, drawing QE parallel to PB, and ED to BA; then will CDE be the triangle which was to be described.

DEMONSTRATION.

The angle DCE is equal to the given angle by construction; also EQ being parallel to BP, DE to AB, and AP = BC, therefore must DQ = EC (*Euc.* 4. 6.), and consequently DC — EC = CQ. Moreover, if CG be supposed to cut DE in *n*, then D*n* : E*n* : : AG : GB. *Q. E. D.*

Method of Calculation.

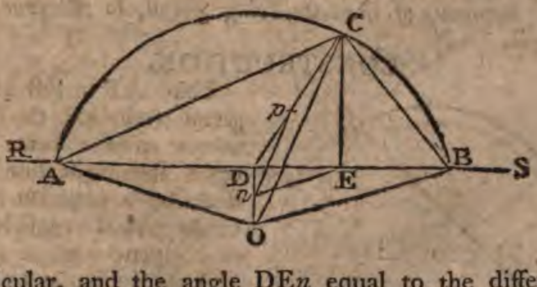
Let C*m* be equal to CE, and let E*m* be drawn. It will be, as AB is to AG — BG, so is the sine of ACB to the sine of the difference of CBA and CAB (*by prob.* 4.); then in the triangle DE*m* will be given all the angles and the side D*m*, whence DE will be given.

PROBLEM X.

The angle at the vertex, the perpendicular, and the difference of the segments of the base, being given, to construct the triangle.

CONSTRUCTION.

Draw RS at pleasure, in which take DE equal to half the difference of the segments of the base, and make EC perpendicular to RS and equal to the given perpendicular, and the angle DE*n* equal to the difference between the given angle and a right one; join D, C, and draw D*n*O parallel to CE, and in DC take the



point *p*, so that *np* (when drawn) may be equal to *n*E; draw CO parallel to *np*, meeting D*n*O in O; and upon O as a center, with the radius OC, describe the circle BCA, cutting RS in B and A; join A, C, and B, C, and the thing is done.

DEMONSTRATION.

Join O, B, and O, A: since OC is parallel to *pn*, therefore is OC : DO : : *pn* : *n*D, or OB : DO : : *n*E : *n*D; and consequently the triangle OBD similar to the triangle *n*ED (*by Euc.* 7. 6.). Therefore, seeing the angle DE*n* is (*by construction*) equal to the excess of the given angle above a right one, ACB must be equal to the angle given (*by prob.* 4.). Moreover, since AD is = DB, AE — BE will be equal to 2DE, which is the given difference of the segments (*by construction*). *Q. E. D.*

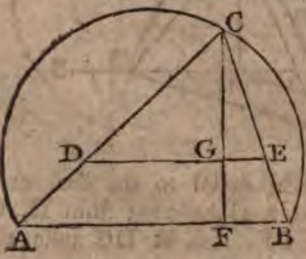
Method of Calculation.

In the triangle CDE, right-angled at E, are given both the legs DE and EC, whence the angle EDC will be known, and consequently ODC; then, as the radius is to the sine of DBO (: : OB : DO : : OC : DO) so is the sine of ODC to the sine of OCD; whence DOC, the difference of the angles ABC, BAC (*see prob.* 4.), is also given, and from thence the angles themselves.

PROBLEM XI.

The angle at the vertex, the perpendicular, and the ratio of the segments of the base, being given, to construct the triangle.

CONSTRUCTION.



Take AF to FB in the given ratio of the segments of the base, and upon the right line AB describe a segment of a circle ACB capable of the given angle; make FC perpendicular to AB, meeting the circumference of the circle in C, in which take CG equal to the given perpendicular; draw DGE parallel to AB, meeting AC and CB in D and E; and then DCE will be the triangle required.

DEMONSTRATION.

Because of the parallel lines DE and AB, it will be as AF : DG (: : CF : CG) : : FB : GE, or AF : FB : : DG : GE; whence it appears, that DG and GE are in the ratio given. Also the angle DCE and the perpendicular CG are respectively equal to the given angle and perpendicular, by construction. *Q. E. D.*

Method of Calculation.

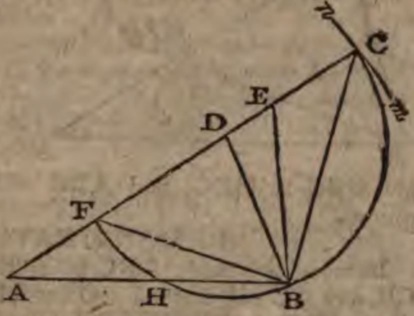
As AB is to AF — BF (*see prob.* 4.), so is the sine of ACB to the sine of the difference of A and B; whence both A and B will be given, because their sum, or the angle at the vertex, is given: then, in the triangles DGC, EGC, will be given all the angles and the perpendicular CG, whence the sides will also be known.

PROBLEM XII.

The base, the sum of the sides, and the difference of the angles at the base, being given, to describe the triangle.

CONSTRUCTION.

At the extremity of the base AB, erect the perpendicular BE, and make the angle EBC equal to half the given difference of the angles at the base; from the point A, to BC, apply AC equal to the sum of the sides; and make the angle CBD = BCA; then will ABD be the triangle required.



DEMONSTRATION.

From the centre D, with the radius CD, describe the semicircle CHF, and join F, B. Then, whereas by construction the angle CBD is = BCD, therefore is DB = DC; whence it appears that AD + DB is = AC, and that the semicircle must pass through the point B: therefore the angle CBF, standing in a semicircle, being a right angle, and therefore = ABE, let FBE, which is common, be taken away, and there will remain ABF = EBC; but DF being equal to DB, it is manifest that ABF (EBC) is equal to half the difference of the angles ABD and DAB. *Q. E. D.*

Method of Calculation.

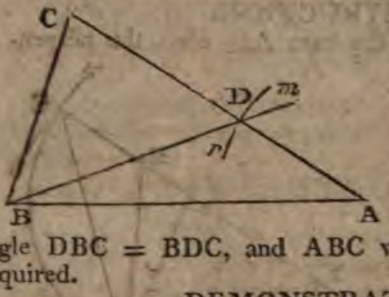
As the sum of the sides (AC) is to the base (AB), so is the sine of ABC, or of the complement of half the given difference, to the sine of (C) half the angle at the vertex; whence the other angles BAD and ABD are also given.

PROBLEM XIII.

The base, the difference of the sides, and the difference of the angles at the base, being given, to determine the triangle.

CONSTRUCTION.

At the extremity B of the given base AB, make the angle ABD equal to half the given difference of the angles at the base; and from A to BD apply AD = the difference of the sides; draw ADC, and make the angle DBC = BDC, and ABC will be the triangle required.



DEMONSTRATION.

Because the angle DBC is = BDC, CD will be = CB, and AC will exceed BC by AD. Moreover, since A + ABD = (CBD) CDB (*Euc.* 32. 1.), therefore is A + 2ABD (= CBD + ABD) = ABC, and consequently ABC — A = 2ABD, equal to the difference given. *Q. E. D.*

Method of Calculation.

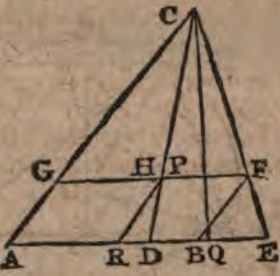
In the triangle ABD are given the two sides, AB and AD, and the angle ABD, whence the angles A and ADB will be given, and from thence the angles CBA and ACB.

PROBLEM XIV.

The difference of the angles at the base, the ratio of the sides, and either the base, the perpendicular, or the difference of the segments of the base, being given, to describe the triangle.

CONSTRUCTION.

Draw AC at pleasure, and make the angle ACD equal to the given difference of the angles at the base, and take CD to CA, in the given ratio of the sides; draw ADE, upon which let fall the perpendicular CQ; take QE equal to QD, and join E, C; then, if the base be given, let AB be taken equal thereto, and draw BF parallel to CA (meeting CE in F), and FG parallel to EA; but if the perpendicular be given, let CP be taken equal thereto, and through P draw FPG parallel to AE; lastly, if the difference of the segments of the base be given, then let AR be taken equal to that difference, draw RH parallel to CA, and FHG to EA; then will CFG be the triangle required.



DEMONSTRATION.

Since $QE = QD$, and the angle $EQC = DQC$, therefore is $CE = CD$, and the angle $E = QDC = A + ACD$ (*Euc.* 32. 1.), and therefore $E - A = ACD$; whence, by reason of the parallel lines AE, GF, &c. we have $GFC - FGC = ACD$, also $FG = AB$, $GH = AR$, and $CF : CG :: CE\ (CD) : CA$. *Q. E. D.*

Method of Calculation.

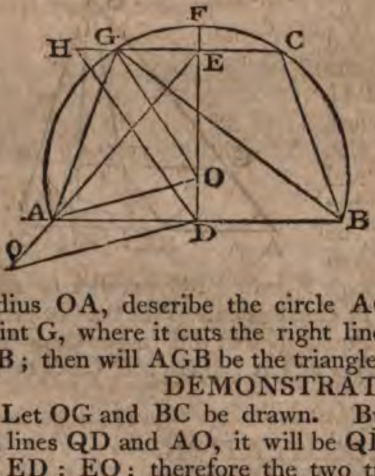
Let CA and CD be expressed by the numbers exhibiting the given ratio of the sides: then in the triangle ACD will be given two sides and the included angle ACD; whence the angles CAE (CGF) and CEA (CFG) will be given, and from thence the sides CG and CF.

PROBLEM XV.

The base, the perpendicular, and the difference of the angles at the base, being given, to construct the triangle.

CONSTRUCTION.

Bisect the given base AB by the perpendicular DF, in which take DE equal to the given height of the triangle; draw CEGH parallel to AB, and make the angle EDH equal to the given difference of the angles at the base; draw EAQ, and take Q therein, so that QD = DH; and, parallel to QD, draw AO, meeting DE in O; upon O, as a center, with the radius OA, describe the circle AGFCB, and from the point G, where it cuts the right line CH, draw GA and GB; then will AGB be the triangle required.



DEMONSTRATION.

Let OG and BC be drawn. By reason of the parallel lines QD and AO, it will be QD (DH) : AO (OG) : : ED : EO; therefore the two triangles EHD, EGO, having one angle, E, common, and the sides about the other angles D and O proportional, are equiangular (*Euc.* 7. 6.), and consequently EOG = EDH. Moreover, because DOEF is perpendicular both to AB and GC, and AD equal to BD, it is evident that the circle passes through the point B, and that the arches FC, FG, as well as the angles ABC, BAG, are equal; and consequently that the angle GBC is the difference of the angles BAG, ABG: but this difference GBC is equal to EOG, or EDH (*Euc.* 20. 3.), that is, equal to the difference given. *Q. E. D.*

Method of Calculation.

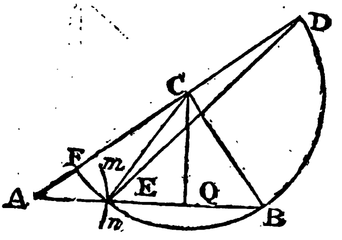
First, in the right-angled triangle AED are given both the legs AD and DE, whence the angle DEA will be given: then it will be, as the radius is to the sine of the angle H, the complement of the given difference (: : DH : DE : : DQ : DE), so is the sine of DEA to the sine of Q; whence AOE (QDE) will also be given; from which take GOE, and there will remain AOG, equal to twice ABG, the lesser angle at the base.

PROBLEM XVI.

The sum of the sides, the difference of the segments of the base, and the difference of the angles at the base, being given, to describe the triangle.

CONSTRUCTION.

Make AD equal to the sum of the sides, and the angle ADE equal to half the difference of the angles at the base; from A to DE apply AE equal to the given difference of the segments of the base; make the angle CED = EDC, and from the point C, where EC cuts AD, with the radius EC, describe the semicircle FEB, cutting AE, produced in B; join B, C, and the thing is done.



DEMONSTRATION.

Upon AB let fall the perpendicular CQ.

Because EQ is = BQ (*Euc.* 3. 3.), therefore will $AQ - BQ = AE$: also, because the angles CED, EDC are equal (*by construction*), CD will be $= CE = CB$, and consequently $AC + CB = AD$. Moreover, ABC

— BAC = BEC — BAC = ACE (*Euc.* 32. 1.) = 2ADE (*Euc.* 20. 3.) *Q. E. D.*

Method of Calculation.

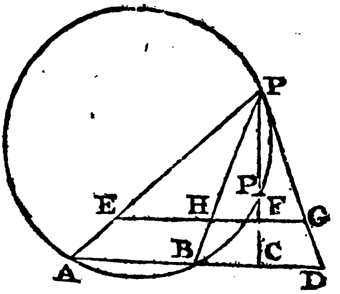
In the triangle ADE are given the sides AD, AE, and the angle D, whence the angle A will be given; then, in the triangle ACE are given all the angles, and the side AE, whence AC and CB (CE) will be likewise given.

PROBLEM XVII.

The difference of the angles at the base, the ratio of the segments of the base, and either the sum of the sides, the difference of the sides, or the perpendicular, being given, to construct the triangle.

CONSTRUCTION.

Let AC be to BC in the given ratio of the segments of the base; and upon AB let a segment of a circle BPA be described (*by problem* 4), to contain an angle equal to the difference of the angles at the base; raise CP perpendicular to AC, cutting the periphery of the circle in P, and in AC produced, take CD = CB, and draw PA, PB and PD: then, if the perpendicular be given, take PF equal thereto, and through F draw EFG parallel to AD; but if the sum or difference of the sides be given, let a fourth proportional PE, to AP ± PD, AP, and the said sum or difference be taken, and draw EFG as above; then will PEG be the triangle required.



DEMONSTRATION.

Since CP is perpendicular to AD, and CD = CB, the angle D will be equal to DBP = A + BPA: whence, because EG is parallel to AD, PGE will be = PEG + BPA (*Euc.* 29. 1.), and consequently PGE — PEG

= ABP, which, by construction, is equal to the given difference of the angles at the base.

Again, by reason of the parallel lines AD and EG, it will be, EF : FG : : AC : (BC) CD. Likewise, for the same reason, AP ± PD : PA : : PE ± PG : PE : : given sum or diff. of sides : PE (*by construction*) and consequently PE ± PG = the said given sum or difference. *Q. E. D.*

Method of Calculation.

First, it will be, as AB is to AD, so is the sine of APB to the sine of APD (*by prob.* 4); then, in the triangle PGE, will be given all the angles, and either the perpendicular, or the sum or difference of the sides, whence the sides themselves are readily determined.

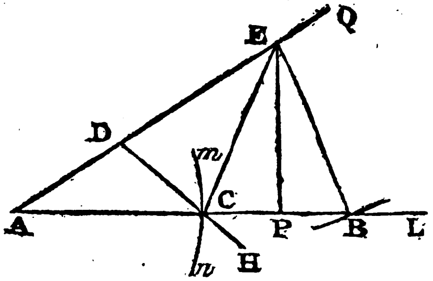
Note. The perpendicular cutting the circle in two points, indicates that this problem is capable of two different solutions.

PROBLEM XVIII.

The difference of the sides, the difference of the segments of the base, and the difference of the angles at the base, being given, to describe the triangle.

CONSTRUCTION.

Draw the indefinite line AQ, in which take AD equal to the given difference of the sides, and make the angle QDH equal to the complement of half the difference of the angles at the base; from A to DH apply AC = the given difference of the segments; and having produced the same to L, make the angle DCE equal to CDE, and let CE meet AQ in E, and upon the center E, with the radius EC, describe an arch, cutting AL in B; join E, B, so shall AEB be the triangle required.



DEMONSTRATION.

Upon AB let fall the perpendicular EP.

Because the angle DCE = CDE, therefore is ED = EC, and consequently AE — EB (= AE — EC = AE — ED) = AD. Also, since EB = EC, therefore will PB = PC, and consequently AP — BP (AP — PC) = AC. Moreover, the angle EBC being = ECB (*Euc.* 5. 1.), and ECB — A = CEA (*Euc.* 32. 1.), it is plain that EBC — A = CEA, equal to the given difference, because the triangle EDC is isosceles, and the angle at the base equal to the complement of half the said difference, by construction. *Q. E. D.*

Method of Calculation.

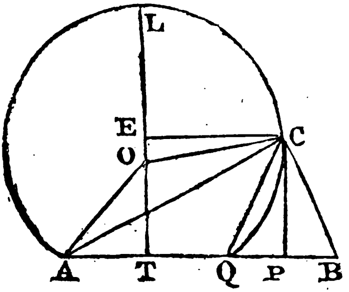
In the triangle ADC are given two sides, and the angle ADC, whence the angle A will be known; then, in the triangle ACE, will be given all the angles and the side AC, whence AE and CE (BE) will also become known.

PROBLEM XIX.

The perpendicular, the difference of the angles at the base, and the difference of the segments of the base, being given, to construct the triangle.

CONSTRUCTION.

Upon AQ, equal to the given difference of the segments of the base, let a segment of a circle QCA be described, capable of the difference of the angles at the base; bisect AQ with the perpendicular TL, in which let TE be taken equal to the given perpendicular; draw EC parallel to AQ, cutting the periphery of the circle in C; also draw CP perpendicular to AQ, and in AQ produced take PB = PQ; join C, A, and C, B; then will ACB be the triangle required.



DEMONSTRATION.

Since (*by construction*) CP is perpendicular to QB, and PB equal to PQ, thence will the angle B = PQC, and B (PQC) — BAC = ACQ = difference of angles given: also, for the same reason, will CP = TE, and AP — BP = AP — PQ = AQ. *Q. E. D.*

Method of Calculation.

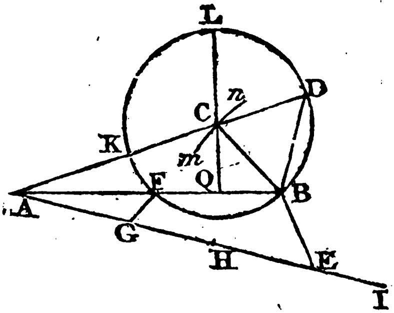
From the center O, conceive OA and OC to be drawn: then, in the triangle AOT, will be given all the angles, and the side AT, whence OT and OE will be given; then it will be, as AT : OE : : sine of AOT (ACQ) : sine of OCE; whence all the angles in the figure are given.

PROBLEM XX.

The segments of the base, and the sum of the sides of any plane triangle, being given, to determine the triangle.

CONSTRUCTION.

From the greater segment AQ, take QF equal to the lesser segment BQ; make QL perpendicular to AB, and draw AI, making any angle with AB at pleasure, in which take AE equal to the given sum of the sides, and join B, E; make the angle AFG = AEB, and bisect EG in H, and from B as a center, with the radius EH, describe mCn, cutting the perpendicular QL in C; join C, A, and C, B, and the thing is done.



DEMONSTRATION.

From the center C, with the radius CB, let the circle BDLKF be described; and let AC be produced to meet its periphery in D. By reason of the similar triangles AEB, AFG, it will be, as AE : AB : : AF : AG; whence AG × AE = AF × AB; but (*by Euc.* 37. 3.)

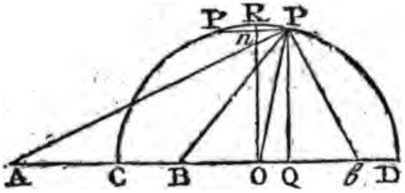
whence the perpendicular PQ ($= \sqrt{CQ \times DQ}$) is likewise given.

PROBLEM XXIII.

Having the base, the perpendicular, and the ratio of the sides, to describe the triangle.

CONSTRUCTION.

Let the base AB be divided at C, in the given ratio of the sides, and let the circle CPD be described as in the last problem; in OR, perpendicular to AD, take O*n* equal to the given perpendicular, and, through *n*, draw P*n*P parallel to AD, cutting the periphery of the circle in P; join P, A, and P, B, and the thing is done. The truth of this is also evident from the preceding lemma.



Method of Calculation.

Upon AD let fall the perpendicular PQ, and join O, P: then PO $\left(= \dfrac{AC \times BC}{AC - BC}\right)$ will be given; therefore, in the triangle OPQ, are given OP and PQ, from whence not only OQ, but AQ and BQ are also given.

Note. The parallel P*n*P, cutting the circle in two points, shows that this problem admits of two different solutions.

PROBLEM XXIV.

The difference of the segments of the base, the perpendicular, and the ratio of the sides, being given, to construct the triangle.

CONSTRUCTION.

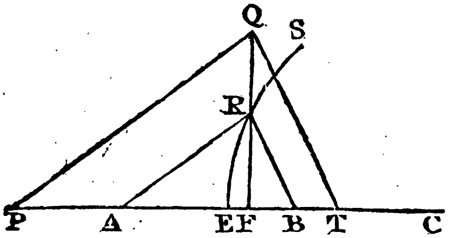
Let AB be the difference of the segments of the base (*see the last figure*), and let every thing be done as in the preceding problem: take Q*b* = QB, and join P, *b*; then will A*b*P be the triangle required. The reasons of which are obvious from what has been said already; and the numerical solution is also evident from the last problem.

PROBLEM XXV.

The ratio of the segments of the base, the perpendicular, and the ratio of the sides being given, to construct the triangle.

CONSTRUCTION.

Draw any right line ABC at pleasure, in which take AE to EB in the given ratio of the sides, and AF to FB in the given ratio of the segments of the base, and make FQ perpendicular to AB, and equal to the given height of the triangle; make also EC : AE : : BE : AE — BE, and with the radius CE describe the circle ERS, and from the point R, where it intersects the perpendicular FQ, draw RA and RB, and draw QP and QT parallel to RA and RB; then will PQT be the triangle that was to be described.



DEMONSTRATION.

By the foregoing lemma, AR : BR : : AE : BE; therefore, by reason of the parallel lines, it will be QP : QT (: : RA : RB) : : AE : BE. And, for the same reason, PF : TF : : AF : BF. *Q. E. D.*

Method of Calculation.

Having assumed AB at pleasure, there will be given BE, AE, BF, and CE $\left(= \frac{AE \times BE}{AE - BE}\right)$; whence RF $\left(= \sqrt{EF \times \overline{CE + CF}}\right)$ is also given; then, in the right-angled triangle BRF, will be given both the legs

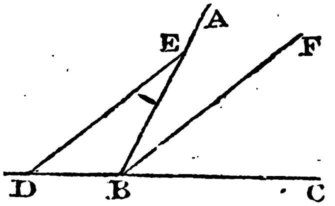
BF and RF, whence the angle B is given; lastly, in the triangle FQT will be given all the angles and the side FQ, whence QT and TF will be given, and consequently PQ and FP.

PROBLEM XXVI.

To divide a given angle ABC *into two parts*, CBF, ABF, *so that their sines may obtain a given ratio.*

CONSTRUCTION.

In BA, and CB produced, take BE and BD in the given ratio of the sine of CBF to the sine of ABF; draw DE, and parallel thereto draw BF, and the thing is done. For, *by trigonometry*, BE : BD : : the sine of D (= CBF) : the sine of BED (= ABF). Hence the numerical solution is also evident; since it will be, as the sum of BE and BD is to their difference, so is the tangent of half the given angle ACB to the tangent of half the difference of the two required parts FBC and FBA.

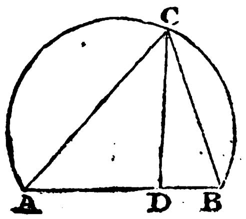


PROBLEM XXVII.

To divide an angle given into two parts, so that their tangents may be to each other in a given ratio.

CONSTRUCTION.

Take any two right lines AD, BD, which are in the ratio given, and upon the whole compounded line AB let a segment of a circle BCA be described, capable of the angle given; make DC perpendicular to AB, meeting the periphery in C, and draw AC and BC, then will ACD and BCD be the two angles required.



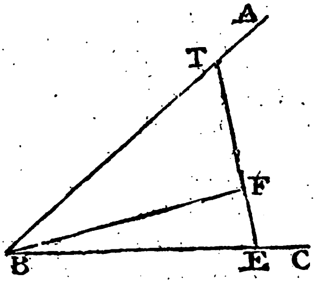
The reason of which is evident, at one view, from the construction. The method of solution is also very easy; for it will be, as AB is to AD — DB, so is the sine of ACB to the sine of B — A (*see problem* 4); whence B and A, and also BCD and ACD, are given.

PROBLEM XXVIII.

To divide a given angle ABC *into two parts, so that their secants may obtain a given ratio.*

CONSTRUCTION.

Take BE to BT in the given ratio of the secants; join T, E, and let BF be drawn perpendicular to ET,



and the thing is done. The truth of which is manifest from the construction.

Method of Calculation.

The angle EBT and the ratio of the sides BE and BT being given, the angles E and T will also be given, and consequently their complements EBF and FBT.

PROBLEM XXIX.

From a given point O, *to draw a right line* OF, *to cut two right lines* AC, AB, *given by position, so that the parts thereof,* OE, OF, *intercepted between that point and those lines, may be to one another in a given ratio.*

CONSTRUCTION.

From O, through A, the point of concourse of BA and CA, let OAD be drawn, in which take AD to AO and therefore AD : FD : : AQ : HQ : : tang. of DHQ : tang. DAQ. Likewise DA : DE (DH) : : sine of DHQ : sine of DAQ, as was to be shown.

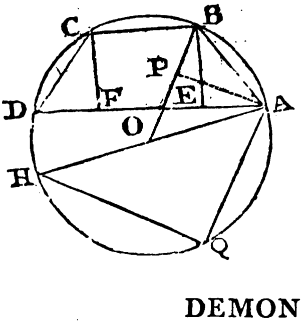
Method of Calculation.

If AR be supposed to meet the periphery in R, and RN be drawn parallel to HF, meeting AK in N; then will DN = DF, and AN : AR : : AF : AH; but (*by Euc.* 37. 3.) AR : AK : : AE : AH; whence, by compounding the terms of these two proportions, &c. AN : AK : : AF × AE : AH^2; whence AH, as well as AD and DH, being known, the angles A and K will also be known.

PROBLEM XXXII.

To draw from a point A, *in the circumference of a given circle, two subtenses* AB *and* AD, *which shall be to one another in the given ratio of m to n, and cut off two arches* AB *and* ABD, *in the ratio of* 1 *to* 3.

CONSTRUCTION.



Draw the diameter AH, and take the subtense AQ, in proportion thereto, as *n* — *m* to 2*m*; from the center O draw OB parallel to AQ, meeting the periphery in B; join A, B, and make the subtenses BC and CD each equal to AB, and draw AD, and the thing is done.

DEMONSTRATION.

Join H, Q, and draw BE and CF perpendicular to AD.

The angle AOB (QAH) at the center, standing upon the arch AB, is equal to the angle BAD at the circumference, standing upon double that arch; therefore, AQH being equal to AEB, or a right angle (*Euc.* 31. 3.), the triangles AQH, AEB, must be equiangular, and consequently AB : AE : : AH : AQ; but, by con-

DEMONSTRATION.

Draw CM and DNE perpendicular to the radius OB, and Nn and Ee perpendicular to DF.

It is evident, by construction, that the triangles OCM and DNn are similar (because Nn is parallel to CO, and ND to CM); therefore OC : CM : : DN : Nn ($=\frac{1}{2}$Ee), and consequently CM × DN = OC × $\frac{1}{2}$Ee = $\frac{1}{2}$OC × Ee = FG × Ee = the given rectangle by construction. *Q. E. D.*

Method of Calculation.

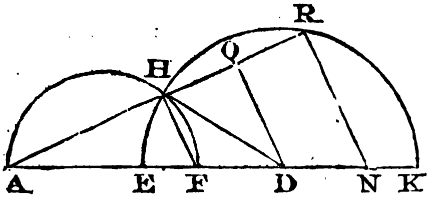
Dividing the measure of the given rectangle by half the radius, FI will be given, which, added to OF, the co-sine of CD, gives the co-sine (OI) of CE, the difference of the two parts; whence the parts themselves will be known.

PROBLEM XXXI.

Having the ratio of the sines, and the ratio of the tangents of two angles, to determine the angles.

CONSTRUCTION.

Let AD be to ED in the given ratio of the sines, and AD to FD in the given ratio of the tangents; and about the center D, with the interval DE, let the semicircle ERK be described; and, upon AF, describe another semicircle, cutting the former in H, and through H draw AR, and join H, D; then will DHR and DAR be the two angles required.



DEMONSTRATION.

Join F, H, and draw DQ perpendicular to AR.

The angle AHF, standing in a semicircle, being a right one, the lines FH and DQ are parallel (*by Euc.* 27. 1.);

DEMONSTRATION.

The triangle ABC, standing upon the whole diameter AB, is equal to the rectangle OE, of the same altitude, standing upon half AB (*by Euc.* 41. 1.), which last (*by construction*) is equal to the area given.

Method of Calculation.

Join O, C, and let CD be perpendicular to AB; then it will be, as AO^2 $(AO \times OC)$: $AO \times DC$ (: : OC : DC) : : radius : sine of DOC; which, in words, gives this theorem.

As the square of half the hypothenuse of any right-angled plane triangle is to the area, so is the radius to the sine of double the lesser of the two acute angles.

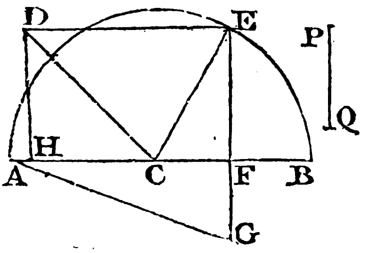
N. B. Since no sine can be greater than the radius, it is plain, that, if the square of half the hypothenuse be not given greater than the area of the triangle, the problem will become impossible; in which case the side EF, instead of cutting, will pass quite above the circle.

PROBLEM XXXIV.

To describe a right-angled triangle, whose area shall be equal to a given square, and the sum of its two legs equal to a given right line AB.

CONSTRUCTION.

Upon AB let a semicircle be described; make ACD = half a right angle, and CD = twice (PQ) the side of the given square; draw DE parallel to AB, meeting the circumference in E, and EF perpendicular to AB, intersecting AB in F, in which, produced, take FG = FB, and draw AG; so shall AFG be the triangle required.



DEMONSTRATION.

It is evident that AF + FG = AB; and also that the area $AFG = \frac{1}{2}AF \times FG = \frac{1}{2}AF \times FB = \frac{1}{2}FE^2$ ($= \frac{1}{2}DH^2$) $= \frac{1}{4}CD^2 = PQ^2$. *Q. E. D.*

Method of Calculation.

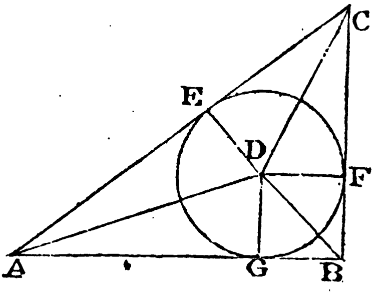
If the radius CE be drawn, in the right-angled triangle CEF, there will be given CE ($= \frac{1}{2}AB$) and $\overline{EF}]^2$ ($= 2PQ^2$) whence CF ($= \sqrt{\frac{1}{4}AB^2 - 2PQ^2}$) will be known, and, from thence, both AF and FG.

LEMMA.

The area of any right-angled triangle, ABC, *is equal to a rectangle under half its perimeter and the excess of that half perimeter above the hypothenuse, or longest side.*

DEMONSTRATION.

In the proposed triangle let the circle EGF be inscribed, and from the center D, to the angular points A, B, C, and the points of contact E, F, G, let the right lines DA, DB, DC, DE, DF, and DG be drawn. It is plain that the sum of the three triangles ADB, BDC, and ADC, is equal to the whole triangle ABC; but the triangle ADB is equal to the rectangle $\frac{1}{2}AB \times DG$; and so of the rest: therefore the sum of the rectangles $\frac{1}{2}AB \times DG + \frac{1}{2}CB \times DF + \frac{1}{2}AC \times DE$ is equal to the whole triangle ABC; but the sum of these rectangles (*by Euc.* 1. 2.) is equal to the rectangle under half the perimeter $AB + BC + AC$ and the semi-diameter DG, which last rectangle is, therefore, equal to the triangle given. But the angles E and G being right ones (*Euc.* 17. 3.), and the side AD common, and also DE equal to DG, thence will $AE = AG$ (*Euc.* 47. 1.). And in the same manner will $CE = CF$; consequently AC ($AE + CE$) will be $= AG + CF$; whence it



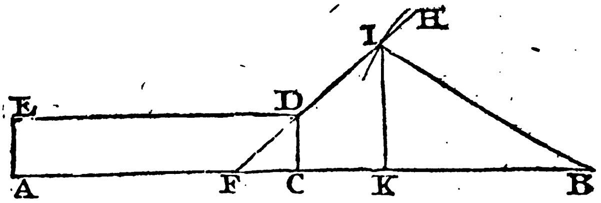
appears that the hypothenuse is less than the sum of the two legs by BG + BF, or twice the radius of the inscribed circle, and therefore less than half the perimeter by once that radius, or DG: whence the proposition is manifest.

PROBLEM XXXV.

The perimeter and area of a right-angled triangle being given, to describe the triangle.

CONSTRUCTION.

Make AB equal to the given perimeter, which bisect in C, and upon AC let the rectangle ACDE be constituted equal to the given area; take CF = CD, and, from F, through D, draw the indefinite line FH, to which, from B, apply BI = AF; then, upon AB let fall the perpendicular IK, so shall BIK be the triangle that was to be constructed.



DEMONSTRATION.

Since (*by construction*) CD is = CF, therefore is IK = FK, and consequently IK + IB + BK = FK + AF + BK = AB. Again, the excess of the half perimeter AC above the hypothenuse BI (AF) being = CF = CD, it is evident (*from the premised lemma*) that the area of the triangle will be = ACDE = the given area by construction. *Q. E. D.*

Method of Calculation.

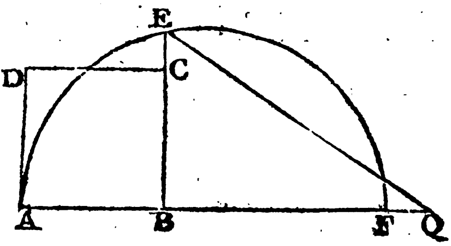
Dividing the area by half the perimeter, CD (= CF) will be given; then, in the triangle BFI, will be given BF, BI, and the angle F (= 45°); whence the angle B will also be known, and from thence BK and BI.

PROBLEM XXXVI.

To make a right-angled triangle equal to a given square **ABCD**, *whose sides shall be in arithmetical progression.*

CONSTRUCTION.

In AB, produced, take BF $= \frac{3AB}{2}$, and upon AF describe the semicircle AEF, cutting BC, produced,



in E; take BQ $= \frac{4EB}{3}$; join E, Q, and the thing is done.

DEMONSTRATION.

Since, by construction, QB : BE : : 4 : 3, therefore will $BQ^2 : EB^2$: : 16 : 9, and $BQ^2 + BE^2 : BE^2$: : 16 + 9 (25) : 9, that is, $EQ^2 : BE^2$: : 25 : 9 (*Euc.* 47. 1.); whence EQ : BE : : 5 : 3 (*Euc.* 22. 6.); therefore the sides BE, BQ, and EQ, being to one another in the ratio of the numbers 3, 4, and 5, are in arithmetical progression. And, because BQ is $= \frac{4EB}{3}$, thence will $\frac{EB \times BQ}{2} = \frac{2EB^2}{3} = \frac{2BF \times AB}{3} = AB^2$. *Q. E. D.*

Method of Calculation.

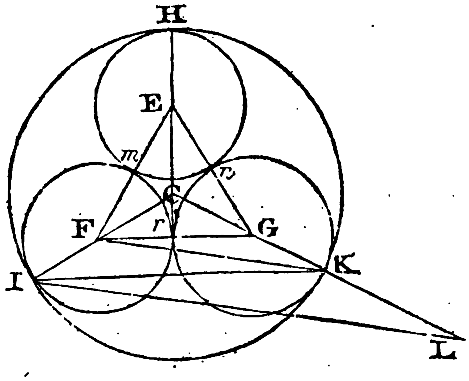
Seeing BF is $= \frac{3AB}{2}$, (BE $\sqrt{AB \times BF}$) will be $= AB\sqrt{\frac{3}{2}}$; whence BQ $\left(\frac{4BE}{3}\right)$ and EQ $\left(\frac{5BE}{3}\right)$ will be likewise given.

PROBLEM XXXVII.

In a given circle CIHK, *to describe three equal circles* E, F, *and* G, *which shall touch one another, and also the periphery of the given circle.*

CONSTRUCTION.

From the center C let the right lines CH, CI, and CK be drawn, dividing the periphery into three equal parts, in the points H, I, and K; join I, K, and in GK, produced, take KL = $\frac{1}{2}$IK; draw IL, and, parallel thereto, draw KF meeting CI in F; make HE and KG each = IF, and upon the centers F, E, and G, through the points I, H, and K, let the circles F*r*I, E*m*H, and G*n*K be described, and the thing is done.



DEMONSTRATION.

Draw FE, FG, and EG.

Because (*by construction*) HE, IF, and KG are equal, CE, CF, and CG will likewise be equal, and FG parallel to IK (*by Euc.* 2. 6.), and therefore, KF being parallel to IL (*by construction*), the triangles IKL and FGK are equiangular; whence, IK being = 2KL, FG is = 2GK (2F*r*) (*Euc.* 4. 6.): whence it is manifest that the circles F and G touch each other.

Moreover, the angles ECF, ECG, and FCG, as well as the containing sides CE, CF, and CG being respectively equal, EF, FG, and EG must also be equal (*by Euc.* 4. 1.), and therefore EF or EG = 2FI or 2GK; whence it is evident that the circles E, F, and G also touch one another. But all these circles touch the given circle, because they pass through given points H, I, K in its periphery, and have their centers in right lines joining the center C and the points of concourse.

Method of Calculation.

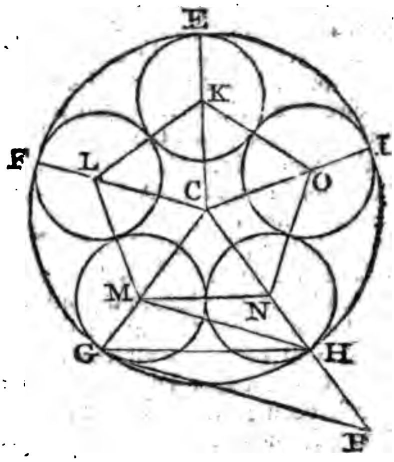
In the triangle FGK we have given the angle FGK (150°), and the ratio of the including sides (*viz.* as 2 to 1); whence the angle FKG will be given; then, in the triangle FCK will be given all the angles and the side CK; whence CF and also FI will be given. But, if you had rather have a general theorem for expressing the ratio of FI to CI, then let EC be produced to meet FG in *r*. Therefore the angle *r*FC being = 30°, C*r* will be $= \frac{1}{2}$CF; whence (*by Euc.* 47. 1.) FI or F*r* $(\sqrt{FC^2 - Cr^2})$ is $= FC \times \sqrt{\frac{3}{4}}$, and therefore CI $= FC + FC\sqrt{\frac{3}{4}}$; consequently CI : FC : : $1 + \sqrt{\frac{3}{4}} : 1$; whence, by division, CI : FI (: : $1 + \sqrt{\frac{3}{4}} : \sqrt{\frac{3}{4}}$) : : $\sqrt{\frac{4}{3}} + 1 : 1$.

PROBLEM XXXVIII.

In a given circle CEHG *to describe five equal circles* K, L, M, N, *and* O, *which shall touch one another, and the circle given.*

CONSTRUCTION.

Let the whole periphery EGH be divided into five equal parts, at the points E, F, G, H, and I (*by Euc.* 11. 2.), and draw CE, CF, CG, CH, and CI; join G, H, and in CH produced take HP = $\frac{1}{2}$GH; draw PG, and parallel thereto draw HM, meeting CG in M; take FL, EK, IO, and HN, each equal to MG, and upon the centers K, L, M, N, and O, let circles be described through the points E, F, G, H, and I, and the thing is done.



The demonstration whereof is evident from the last proposition: and in the same manner may 6, 8, or 10, &c. equal circles be inscribed in a given circle, to touch one another.

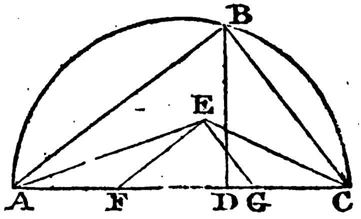
The method of calculation in this, or in any other case, will also be the same as in the last problem; for in the triangle MNH will be given the ratio of NM to NH (as 2 to 1), and the included angle MNH equal to 126°, 120°, 112½°, or 108°, &c. according as the number of circles is 5, 6, 8, or 10, &c. from which the angle MHN will be given; then in the triangle CMH will be given all the angles, and the side CH, to find CM.

PROBLEM XXXIX.

The perimeter of a right-angled triangle, whose sides are in geometrical progression, being given, to describe the triangle.

CONSTRUCTION.

Upon AC, equal to the given perimeter, describe the semicircle ABC, and let AC be divided in D, according to extreme and mean proportion: make DB perpendicular to AC, meeting the periphery of the circle in B, and, having joined A, B, and C, B, let AE and CE be drawn to bisect the angles BAC, BCA; and, from point of intersection E, let EF and EG be drawn parallel to BA and BC, cutting AC in F and G; then will EFG be the triangle that was to be constructed.



DEMONSTRATION.

Since (*by construction*) AC : AD : : AD : DC; therefore is AC*q* : AC × AD : : AC × AD : AC × DC (*by Euc.* 1. 6.), or AC*q* : AB*q* : : AB*q* : BC*q* (*by cor. to Euc.* 8. 6.), and consequently AC : AB : : AB : BC; whence, the triangles ABC, FEG, being equiangular, FG : FE : : FE : EG. Also, EF is = AF, because the angle FEA (= EAB) = FAE; and in the very same manner is EG = GC; therefore EF + FG + EG (= AF + FG + GC) = AC. Moreover, the angle FEG (= ABC) is a right angle, *by Euc.* 31. 3. *Q. E. D.*

Method of Calculation.

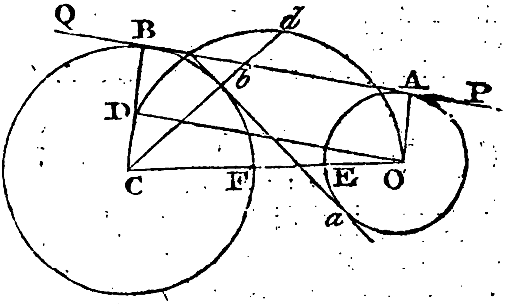
Because (*by construction*) AD ($= \sqrt{\frac{5}{4}ACq} - \frac{1}{2}AC$) = AC × $\sqrt{\frac{5}{4} - \frac{1}{2}}$, thence is AB ($\sqrt{AC \times AD}$) = AC × $\sqrt{\sqrt{\frac{5}{4}} - \frac{1}{2}}$, and BC ($\sqrt{AC \times CD}$ = AD) = AC × $\sqrt{\frac{5}{4}} - \frac{1}{2}$: but, by reason of the similar triangles ABC, FEG, it will be, as AC + AB + BC : (FG + FE + EG) AC : : AC : FG : : AB : FE : : BC : EG; or as $\sqrt{\sqrt{\frac{5}{4}} - \frac{1}{2}} + \frac{1}{2} + \sqrt{\frac{5}{4}}$: 1 : : AC : FG : : AB : FE : : BC : EG; whence FG, FE, and EG are given.

PROBLEM XL.

To draw a right line PQ, *to touch two circles* C *and* O, *given in magnitude and position.*

CONSTRUCTION.

Upon the line CO, joining the centers of the given circles, describe the semicircle CDO, in which inscribe CD equal to the difference of the semi-diameters CF and



OE; and from the point B, where CD produced meets the periphery BF, draw PB perpendicular to CB; then will BP touch both the circles.

DEMONSTRATION.

Join O, D, and draw OA perpendicular to PQ.

The angle CDO, standing in a semicircle, is right; therefore the angles B and A being both right ones, by construction, the angle AOD must also be right, and the figure DOAB a rectangle, and consequently AO = BD = BC — CD = CF — CD = OE (*by construction*). Wherefore, seeing CB and OA are respectively equal to CF and OE, and both the angles A and B right ones, it is evident that the right line PQ touches both the circles. *Q. E. D.*

The numerical solution of this problem is extremely easy; for since the two sides CO and CD of the right-angled triangle CDO are both given, the angles DCO and AOC, determining the points of contact B and A, are from thence given, at one operation.

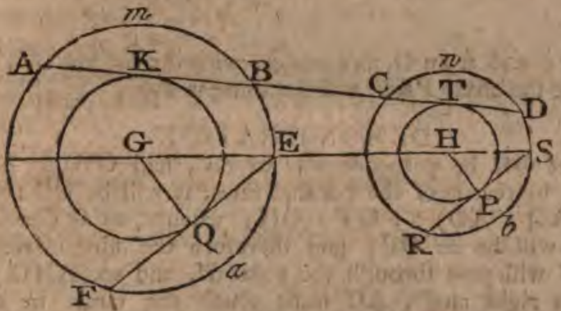
But if it be required to draw a right line (*ab*) to touch both circles, and to pass between the centers C and O; then, instead of taking CD equal to the difference of the semi-diameters CF, OE, let C*d* be taken equal to their sum, and the rest of the process will be exactly the same.

PROBLEM XLI.

To draw a right line AD *through two circles* GAEF, HCSR, *given in magnitude and position, so as to cut off segments thereof,* AKB*m*, CTD*n*, *equal respectively to two given segments* EQF*a*, SPR*b*.

CONSTRUCTION.

Upon the subtenses EF, SR, from the centers G and H, let fall the perpendiculars GQ and HP; and from the same centers, at the distances GQ, HP, let two circles GQK, HPT be described; then draw a right line AD to touch both these circles, by the last proposition, and the thing is done; for the lines FE, AB being at the same distance from the center G, the segments cut off by them must consequently be equal: and, in like manner, the segments SPR*b*, CTD*n*, are also equal.

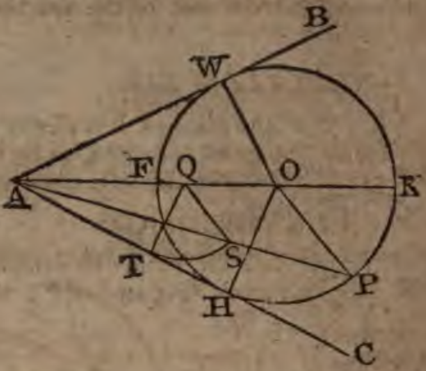


PROBLEM XLII.

To describe the circumference of a circle through a given point P, *to touch two right lines* AB, AC, *given in position.*

CONSTRUCTION.

Join A, P, and bisect the angle BAC, with the right line AK, and, from any point Q in that line, draw QT perpendicular to AC; then, from Q to AP, draw QS = QT; draw, likewiſe, PO parallel to SQ, meeting AK



in O; and from O, as a center, with the radius OP, deſcribe the circle PKF, and the thing is done.

DEMONSTRATION.

Let OH be perpendicular to AC, and OW to AB; then, by reaſon of the parallel lines, it will be QS : OP (: : AQ : AO) : : QT : OH; whence, as QT = QS, OH will be = OP; and therefore the circumference PKF will paſs through the point H, and ſo, AHO being a right angle, AC muſt touch the circle in that point. Moreover, the triangles AOH and AOW being equiangular, and having one ſide common, OW will therefore be = OH, and the circle alſo touch AB in the point W. *Q. E. D.*

Method of Calculation.

Having aſſumed AQ at pleaſure, there will be given, in the triangle AQT, all the angles and one ſide, whence QT (= QS) will be given: then, in the triangle AQS, will be given AQ, QS, and the angle QAS, whence the angle AQS (= AOP) will be given. Laſtly, in the triangle AOP will be given all the angles and the ſide AP, whence AO and PO will be given.

Otherwiſe.

Say, as the ſine of OAH : radius (: : OH : OA : :

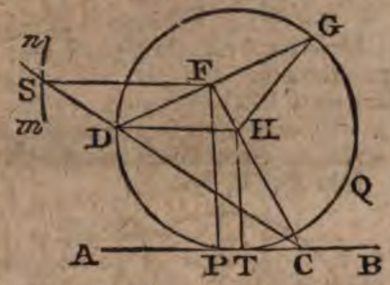
OP : OA) : : sine of OAP : sine of OPA; then, in the triangle AOP will be given all the angles and the side AP; whence the other sides AO and OP will be found.

PROBLEM XLIII.

To describe the circumference of a circle through two given points D, G, *to touch a right line* AB, *given in position.*

CONSTRUCTION.

Draw DG, and bisect the same by the perpendicular FC, meeting AB in C; join C, D, and make FP perpendicular to AB; and, from F to CD, produced, draw FS = FP; make DH parallel to FS, and from H, the intersection of CF and DH, with the radius DH, describe the circle HDQ, and the thing is done.



DEMONSTRATION.

Join H, G, and draw HT parallel to FP, meeting AB in T; then, because of the parallel lines, it will be, FS : HD (: : CF : CH) : : FP : HT; wherefore, as the antecedents FS and FP are equal, the consequents HD and HT must likewise be equal; and, therefore, since HT is perpendicular to AB, the circumference of the circle will touch AB in T; and it will also pass through the point G, because the two triangles DFH, GFH, having two sides and the included angles respeetively equal, are equal in every respect. *Q. E. D.*

Method of Calculation.

The angle FCA, and the numbers expressing FC and DG being given, in the triangle CFD will be given (besides the right angle) both the legs CF and FD, whence CD and the angle FCD will be known;

then it will be, as the sine of FCA (TCH) : radius (: : TH : CH : : DH : CH) : : sine of HCD : the sine of CDH; therefore in the triangle HCD there will be given all the angles and the side CD; whence CH and HD will be known.

PROBLEM XLIV.

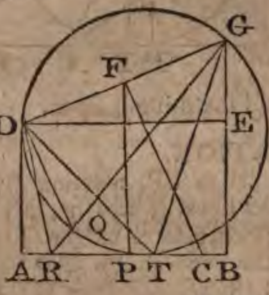
Having given AB, *and also* AD *and* BG, *perpendicular to* AB; *to find a point* T *in* AB, *to which, if two right lines* DT, GT *be drawn, the angle* DTG, *formed by those lines, shall be the greatest possible.*

CONSTRUCTION.

Describe, by the last problem, a circle GDQ, that shall pass through G and D, and touch AB, and the point of contact T will be the point required.

DEMONSTRATION.

Join G, T, and D, T; and from any other point R, in the line AB, draw RG and RD; also, from the point Q, where GR cuts the circle, draw QD; then, the angle GQD, being external with regard to the triangle DQR, will be greater than GRD; therefore GTD, standing in the same segment with GQD, will be also greater than GRD. *Q. E. D.*



Method of Calculation.

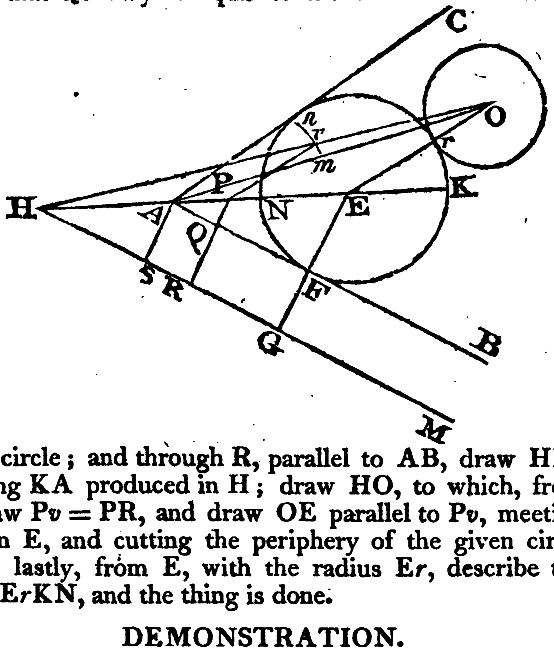
Draw DE parallel to AB; then in the triangle GDE will be given DE, EG (= BG — AD), and the right angle DEG, whence the other angles EDG, EGD, and the side DG will be found; then in the triangle CFP, similar to GDE, will be given all the angles and the side FP $\left(=\frac{AD+BG}{2}\right)$, whence FC will be given; from which, by proceeding as in the last problem, all the rest will be found.

PROBLEM XLV.

To describe a circle, which shall touch two right lines AB, AC, *given in position, and also another circle* O, *given in magnitude and position.*

CONSTRUCTION.

Let the angle CAB, made by the concourse of the two lines, be bisected by AK; and, from any point P in this line, let fall PQ perpendicular to AB, which produce to R, so that QR may be equal to the semi-diameter of the given circle; and through R, parallel to AB, draw HM, meeting KA produced in H; draw HO, to which, from P, draw Pv = PR, and draw OE parallel to Pv, meeting AK in E, and cutting the periphery of the given circle in r; lastly, from E, with the radius Er, describe the circle ErKN, and the thing is done.



DEMONSTRATION.

Draw EG perpendicular to HM, cutting AB in F; then, by reason of the parallel lines, PR : EG (: : HP : HE) : : Pv : EO; therefore, PR being = Pv (*by construction*), EG and EO must likewise be equal; from which the equal quantities FG and Or being taken away, the remainders EF and Er will be equal; and *prob.* 26) by saying, as the sum of the said sines is to their difference, so is the tangent of half BAC to the tangent of half $\overline{BAQ - CAQ}$.

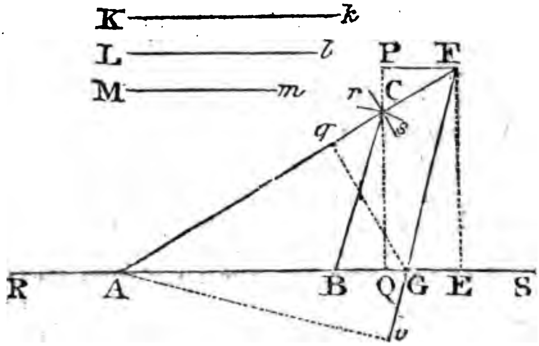
Again, it will be as sine QAH : sine QHA (: : QH : QA : : QP : QA) : : sine QAP : sine QPA; therefore, in the triangle AQP, are given all the angles and one side AP, whence AQ and PQ will be found.

PROBLEM XLVII.

Having the three perpendiculars, let fall from the angles of a plane triangle on the opposite sides, equal to three given right lines K*k*, L*l*, *and* M*m*, *to describe the triangle.*

CONSTRUCTION.

Draw the indefinite right line RS, in which take AB equal to K*k*: find a fourth proportional to M*m*, L*l*, and K*k*, with which, as a radius, from the center A, let an arch *r*C*s* be described; and from B, with the radius L*l*, let another arch be described intersecting the former in C; join A, C, and B, C, and upon RS let fall the perpendicular QC, in which, produced, take QP = L*l*, and draw PF parallel to RS, meeting AC, produced, in F, draw FG parallel to CB, and AFG will be the triangle required.



DEMONSTRATION.

Draw FE, G*q*, and A*v* perpendicular to the three sides of the triangle.

The triangles ABC, AGF; AFE, AGq; and GFE, AGv, are equiangular, *by construction;* therefore Gq : FE : : AG : AF : : AB (Kk) : AC $\left(\frac{\mathrm{K}k \times \mathrm{L}l}{\mathrm{M}m}\right)$: : Mm : Ll; whence, as the consequents FE and Ll are equal, *by construction*, the antecedents Gq and Mm must be equal likewise. Again, BC (Ll) : AB (Kk) (: : FG : AG) : : FE (Ll) : Av; and consequently Kk = Av. *Q. E. D.*

Method of Calculation.

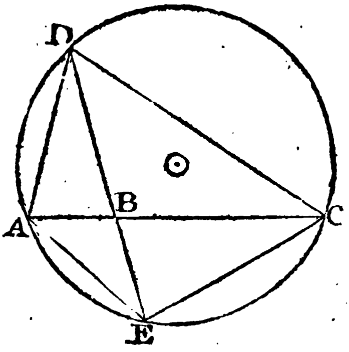
Since Kk, Ll, and Mm are given, AC $\left(=\frac{\mathrm{K}k \times \mathrm{L}l}{\mathrm{M}m}\right)$ will be known; then in the triangle ABC will be given all the three sides, whence the angles are known; lastly, in the triangle AFG will be given all the angles and the perpendicular EF, whence the sides are also known.

PROBLEM XLVIII.

The position of three points, in the same right line, being given, it is proposed to find a fourth, where lines, drawn from the former three, shall make given angles with each other.

CONSTRUCTION.

Let the three given points be A, B, and C: make the angles ACE and CAE respectively equal to the given angles which the lines drawn from B, A, and B, C are to make; and let AE and CE meet in E; through A, C, and E, let the circumference of a circle AECD be described, and, through E and B, draw EBD, meeting it in D, then will D be the point required.



DEMONSTRATION.

Join A, D, and C, D.

The angle EDA is equal to ACE, standing on the same segment; and for the like reason is EDC = CAE. *Q. E. D.*

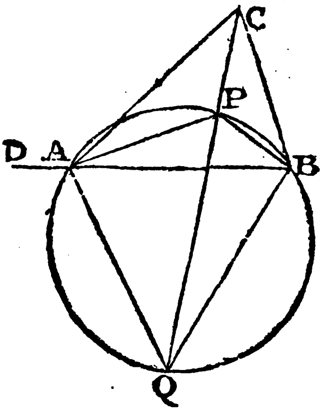
Method of Calculation.

In the triangle ACE are given all the angles and the side AC, whence AE will be given; then in the triangle ABE will be given the two sides AE, AB, and the included angle; whence ABE and all the rest of the angles in the figure will be given.

PROBLEM XLIX.

Three points, A, B, C, *being any how given; to find a fourth, where lines, drawn from the former three, shall make given angles with one another.*

CONSTRUCTION.



Join the given points, and upon the right line AB describe a segment of a circle, capable of the given angle which that line is to subtend; complete the circle, produce BA, and make the angle DAQ equal to the angle which BC is to subtend, and let AQ meet the periphery in Q; draw QC, cutting the same periphery in P; join A, P, and B, P, and the thing is done.

DEMONSTRATION.

The angle ABP is equal to the given angle which AB was to subtend (*by construction*); and the angles QAB and QPA, standing upon the same segment, being equal to each other, their supplements DAQ and BPC must likewise be equal. *Q. E. D.*

Method of Calculation.

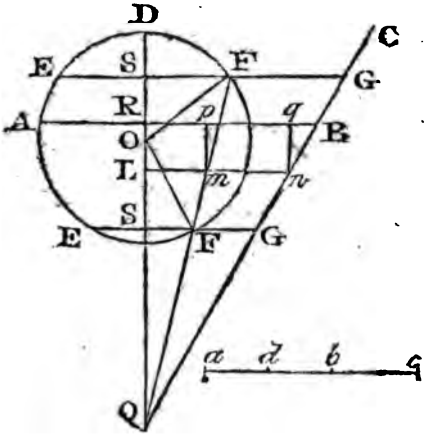
Join B, Q; then, in the triangle ABQ will be given all the angles and the side AB, whence BQ and ABQ will be known; then in the triangle CBQ will be given two sides, and the included angle CBQ; whence the angle CQB, equal to BAP, will be known; lastly, in the triangle APB will be given all the angles and the side AB, from which AP and BP will be found.

PROBLEM L.

To draw a right line EG *through a circle* O, *given in magnitude and position, which shall also cut a right line* QC, *given in position, in a given angle, and have its parts* EF, FG *intercepted by the circle and that right line, in the given ratio of the two right lines ab and bc.*

CONSTRUCTION.

At any point B, in the right line QC, make the angle QBA equal to the given angle, and through the center O, perpendicular to BA, draw DQ meeting BA in R, and CG in Q; bisect *ab* in *d*, and in RB take R*p* = *bd*, and *pq* = *bc*, and draw *pm* and *qn* parallel to DQ; from the point *n*, where *qn* intersects QC, draw *n*L parallel to BA, meeting *pm* in *m*; through the points Q and *m* draw Q*m*F, cutting the periphery of the circle in F, and through F, parallel to BA, draw EFG, and the thing is done.



DEMONSTRATION.

The lines GE, BA, and *n*L, being parallel, the angles QGE, QBA, *&c.* will be equal, and likewise SF : FG : : L*m* : *mn*; but L*m* (*by construction*) is (= R*p*) = *db*, and *mn* (= *pq*) = *bc*; therefore SF : FG : : *db* : *bc*, and consequently EF (2SF) : FG : : *ab* (2*bd*) : *bc*. *Q. E. D.*

Method of Calculation.

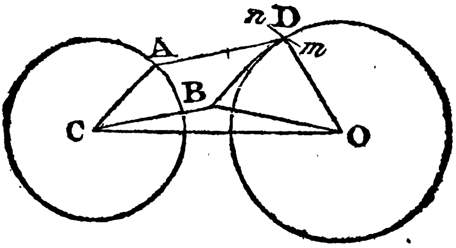
L*n* (*dc*) : L*m* (*db*) : : the tangent of LQ*n* (the complement of the given angle QBR) : the tangent of LQ*m*; therefore, in the triangle OQF will be given one angle OQF and two sides, QO, FO; whence, not only the angle SOF, but also SO and SF will be known.

PROBLEM LI.

To apply, or inscribe, a given right line AD *between the peripheries of two circles* C *and* O, *given in magnitude and position, so as to be inclined to the right line* CO, *joining the centers, in a given angle.*

CONSTRUCTION.

Make OCB equal to the given angle, and let CB be taken equal to the given line; upon the center B, with the radius of the circle C, let the arch *n*D*m* be described, cutting the circle O in D: then draw BD, and, parallel thereto, draw CA, meeting the periphery in A; join A, D, and the thing is done.



DEMONSTRATION.

Because (*by construction*) CA and BD are equal and parallel, therefore will AD and CB be also equal and parallel (*by Euc.* 33. 1.). *Q. E. D.*

Method of Calculation.

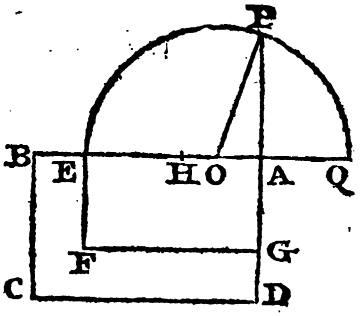
In the triangle CBO are given two sides, CO and CB, and the angle OCB; whence OB and the angle COB will be known; then in the triangle OBD will be given all the three sides, whence all the angles, and consequently DOC, will also be known.

PROBLEM LII.

From a given rectangle ABCD, *to cut off a gnomon* ECG, *whose breadth shall be every where the same, and whose area shall be just half that of the rectangle.*

CONSTRUCTION.

In BA take BH equal to BC, or AD; and in DA, produced, take AP a mean proportional between BA and $\frac{1}{2}$AD (so that AP^2 may = the given area AGFE). From P to the middle of AH draw PO; make OE = OP, and DG = BE; complete the rectangle EAGF, and the thing is done.



DEMONSTRATION.

If the semicircle EPQ, from the center O, be described, it is plain that $AQ = EH = BH - BE = AD - DG = AG$; and consequently that $AE \times AG = AE \times AQ = AP^2$ (*Euc.* 13. 6.) *Q. E. D.*

Method of Calculation.

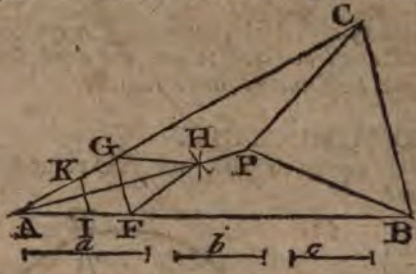
In the right-angled triangle AOP are given AO $\left(= \frac{AB - BC}{2}\right)$ and AP $\left(= \sqrt{\frac{1}{2}AB \times BC}\right)$; whence OP will be known, and from thence both AE and AG.

PROBLEM LIII.

Three points, A, B, C, *being given, it is proposed to find a fourth,* P, *from whence lines, drawn to the three former, shall obtain the ratio of three given lines* a, b, *and* c, *respectively.*

CONSTRUCTION.

Having joined the given points, take AF, in AB, equal to a, and AI $= c$; also make the angles AFG and AIK equal, each, to ACB; and from the centers F and G, with the radii b and AK respectively, let two arcs be described intersecting in H; from which point draw HF and HA; then draw BP to make the angle ABP $=$ AHF, and it will meet AH (produced) in the point P, required.



DEMONSTRATION.

Let BP, CP, and GH be drawn. The triangles ABP, AHF being equiangular (*by construction*), it will be AP : BP : : AF (a) : FH (b); also AB : AP : : AH : AF; and AB : AC : : AG : AF (because ABC and AGF are likewise equiangular); whence it is evident, since the extremes of the two last proportions are the same, that AP × AH = AC × AG, or AC : AP : : AH : AG; therefore the triangles ACP, AHG being equiangular (*Euc.* 6. 6.), we have AP : CP : : AG : GH (AK) : : AF (a) : AI (c). *Q. E. D.*

Method of Calculation.

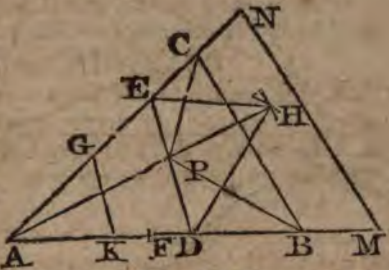
In the triangles AFG, AIK are given all the angles and the sides AF and AI, whence AG, FG, and AK (GH) will be found; then in the triangle FGH will be given all the sides, to find the angle HFG; which, added to AFG, gives AFH (APB), from whence, and the two given sides AF and FH including it, every thing else is readily determined.

PROBLEM LIV.

To describe a triangle (ABC) *similar to a given one* **AMN**, *such that three lines* (AP, BP, CP) *may be drawn from its angular points to meet the same point* (P), *so as to be equal to three given lines* AD, AF, *and* AK, *respectively.*

CONSTRUCTION.

Draw DE and KG, making the angles ADE and AKG, each, equal to the given angle N, and intersecting AN in E and G; from the centers D and E, with the intervals AF and AG, let two arcs be described, intersecting in H; draw AH, in which take AP = AD; and from P, to AM and AN, apply PB and PC equal, respectively, to AF and AK, and let B, C, be joined; so shall ABC be the triangle that was to be determined.



DEMONSTRATION.

The three lines AP, BP, CP are, respectively, equal to the three given lines AD, AF, AK, by construction; we therefore have only to prove that the triangle ABC is similar to the given one AMN. Now, supposing DH and EH to be drawn, it will be AP : PC (or AD : AK) : : AE : AG (EH); whence the triangles APC and AHE will be equiangular (*Euc.* 6. 6.), and consequently AC : AH : : AP (AD) : AE : : AN : AM (*Euc.* 5. 6.); but the triangles ABP and ADH (having AP = AD, PB = DH (*by construction*), and the angle DAP common) are equal in all respects; therefore, by substituting AB in the room of AH, our last proportion becomes AC : AB : : AN : AM; whence it is manifest that the triangles ABC and AMN are equiangular. *Q. E. D.*

Method of Calculation.

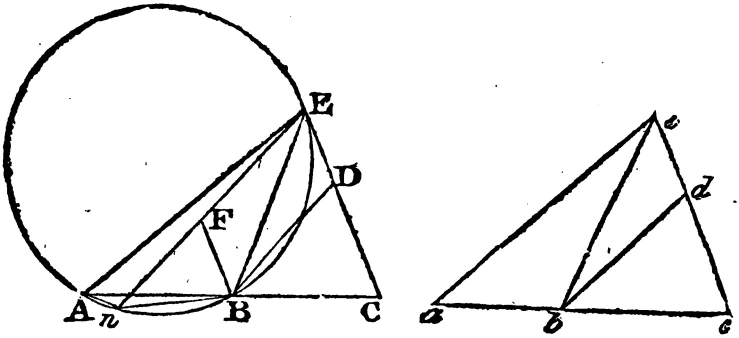
In the triangles ADE, AKG are given all the angles and the sides AD and AK, from which AE, DE, and AG will be known; then in the triangle DHE will be given all the sides, to find the angle EDH, which added to ADE gives ADH; from whence, and the two given sides including it, AH (= AB) will be known.

PROBLEM LV.

In the triangle ace, besides the angle c, are given the segments of the sides ab and de, and the angles aeb and dbe subtended thereby; to describe the triangle.

CONSTRUCTION.

Upon AB, equal to *ab*, let a segment of a circle be described to contain an angle equal to *aeb*: make the angle ABF = *ace*, BA*n* = *dbe*, and the line BF = *ed*; from the point *n*, where A*n* cuts the periphery of the circle, through F, draw *n*FE, meeting the periphery in E; join A, E, and B, E, and draw EC parallel to BF, meeting AB, produced, in C; and then the thing is done.



DEMONSTRATION.

Let BD be parallel to FE.

Since the lines BD, EF, and ED, FB are parallel, therefore is ED = BF (= *ed*), and the angle ACE also = ABF (*ace*) *Euc.* 28. 1. Moreover, the angle BE*n*

(DBE) is equal to BAn (*dbe*), both standing upon the same segment Bn. Q. E. D.

Method of Calculation.

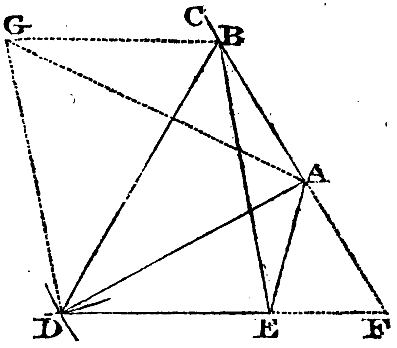
Join B, n; then in the triangle ABn will be given all the angles and the side AB, whence Bn will be known; then in the triangle nBF will be given Bn, BF, and the included angle nBF, whence BFn (CDB) and all the rest of the angles in the figure will be known.

PROBLEM LVI.

To make a trapezium, whose diagonals, and two opposite sides, shall be all of given lengths, and whereof the angle formed by the given sides, when produced till they meet, shall also be given.

CONSTRUCTION.

Draw the indefinite right line AC, and take therein AB equal to one of the two given sides; make the angle CBG equal to the given angle, and let BG be made equal to the other given side; upon the centers A and G, with intervals equal to the two diagonals, let two arches be described, cutting each other in D; make DE equal and parallel to GB; join D, B, and E, A: then ABDE will be the trapezium required.



DEMONSTRATION.

Draw DG, DA, and BE, and let BA and DE be produced to meet each other in F.

The lines BG and DE are equal and parallel, by construction; therefore BE is = DG, which last (by *construction*) is equal to one of the given diagonals, as AD is equal to the other; moreover, the sides AB and ED (BG) are equal to the given sides, by construction; and the angle F is equal to the given angle CBG, because DF is parallel to GB. *Q. E. D.*

Method of Calculation.

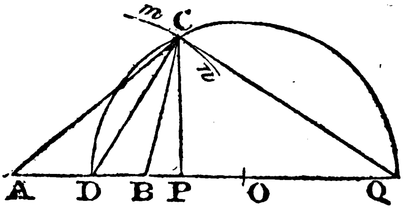
Suppose AG to be drawn; then in the triangle ABG will be given the two sides BA and BG, and the included angle ABG; whence the side AG and the other two angles will be known; then in the triangle ADG will be given all the sides; whence the angle AGD will be known, and from thence the whole angle BGD; lastly, in the triangle BGD will be given the two sides BG and GD, and the included angle BGD; whence the side BD will likewise be known.

PROBLEM LVII.

The segments of the base AD, DB, *and the line* DC, *bisecting the vertical angle* ACB *of a plane triangle, being given, to describe the triangle.*

CONSTRUCTION.

In AB, produced, take DO to AD, as DB to AD — DB, and from the center O, with the radius OD, describe the circle DCQ; also from the center D, at the given distance DC, describe the circle mCn, and from C, the intersection of the two circles, draw CA and CB, and the thing is done.



DEMONSTRATION.

Since DO : AD : : DB : AD — DB; therefore (*by the lemma in* p. 334.) AC : CB : : AD : DB; whence CD bisects the angle ACB (*by Euc.* 3. 6.). *Q. E. D.*

Method of Calculation.

Draw CP perpendicular to AQ.

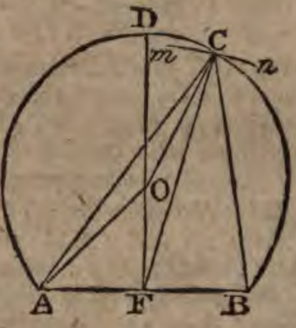
Because, by construction, OD is $= \frac{AD \times BD}{AD - BD}$; therefore will $DQ = \frac{2AD \times BD}{AD - BD}$; whence, by reason of the similar triangles DCQ, DPC, it will be, as $\frac{2AD \times BD}{AD - BD} : DC :: DC : DP = \frac{\overline{AD - BD} \times DC^2}{2AD \times BD}$; whence AC and CB are given.

PROBLEM LVIII.

Having given the base, the angle at the vertex, and the line drawn from thence to bisect the base; to construct the triangle.

CONSTRUCTION.

Upon the given base AB describe (*by prob.* 4.) a segment of a circle ADB capable of the given angle; and, from the point F, in which the perpendicular DF bisects AB, with a radius FC equal to the bisecting line, describe nCm, cutting the periphery ACB in C; join A, C, and B, C, and the thing is done.



The demonstration of which is evident from the construction.

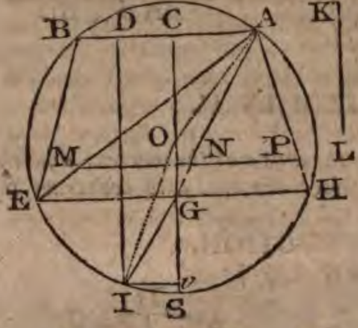
Method of Calculation.

From the center O let OA and OC be drawn; then in the triangle AOF will be given all the angles and the side AF; whence FO and OC (OA) will be known, and in the triangle CFO will be given all the sides; whence the angle FOC, and its supplement DOC, expressing the difference of the angles at the base, will also be known.

PROBLEM LIX.

The base, the difference of the angles at the base, and the line drawn from the vertical angle to bisect the base of any plane triangle, being given; to describe the triangle.

CONSTRUCTION.



Upon AB, equal to the given base, let a segment of a circle AHEB be described to contain an angle equal to the difference of the angles at the base; bisect AB in C, and take CD to AC in the duplicate ratio of AC to the given bisecting line KL; make CS and DI perpendicular to AB, cutting the circle in S and I; draw AI, cutting CS in G; and through G draw the chord EGH parallel to AB; join A, E, and A, H, and in AI take AN equal to KL; draw MNP parallel to EH, meeting AE and AH in M and P; then will AMP be the triangle which was to be constructed.

DEMONSTRATION.

Since (*by construction*) CG is parallel to DI, and $KLq : ACq :: AC : CD$; therefore (*Euc.* 4. 6.) $KLq : ACq :: AG : GI :: AGq : GI \times AG$; but $GI \times AG = EG \times GH = EGq$ (*Euc.* 35. 3. *and* 3. 3.); therefore $KLq : ACq :: AGq : EGq$; and consequently $KL : AC :: AG : EG :: AN : NM$; but AN is (*by construction*) equal to KL, therefore NM is $= AC$, and consequently $MP\ (2MN) = AB$. Moreover, the difference of the angles at the base, $P - M$, is $(= AHE - AEH) = AEB$; which (*by construction*) is equal to the difference given. *Q. E. D.*

Method of Calculation.

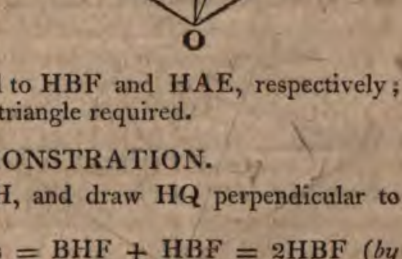
From the center O draw OA and OI; also draw Iv parallel to EH, meeting OS in v: then it will be (*by* *construction*) as KLq : ACq (: : AC : Iv) : : the sine of AOC or AEB, the given difference of the angles at the base, to the sine of SOI, which, added to AOS, gives AOI, whose supplement, divided by 2, will be OIG; from whence OGI and its supplement OGA are given; and consequently ANM (equal to AGE); then in the triangle ANM will be given AN, NM, and the included angle ANM, whence the angles M, A, P will also be given.

PROBLEM LX.

The perpendicular, the angle at the vertex, and the sum of the three sides of a triangle being given; to describe the triangle.

CONSTRUCTION.

Make AB equal to the sum of the sides, which bisect in P, making PO perpendicular to AB, and the angle PAO equal to half the given angle at the vertex; from the center O, with the radius OA describe the circle AHB, and in OP, produced, take PK equal to the given perpendicular, and draw KH parallel to BA, cutting the circle in H; join A, H, and B, H, and make the angles BHF and AHE equal to HBF and HAE, respectively; then will EHF be the triangle required.



DEMONSTRATION.

Join O, B, and O, H, and draw HQ perpendicular to AB.

The angle EFH is = BHF + HBF = 2HBF (*by construction*) = HOA (*Euc.* 20. 3.): and, in the same manner, is FEH = HOB; hence it follows that EFH + FEH (= HOA + HOB) = AOB; and, by taking each of these equal quantities from two right angles, we have EHF = OAB + OBA (*Euc.* 32. 1.) = 2OAB = the given angle (*by construction*). Moreover, QH is = PK = the given perpendicular; and, EH being = AE, and FH = BF (*Euc.* 6. 1.), EH + HF + EF will therefore be = AB = the given sum of the sides. *Q. E. D.*

Method of Calculation.

In the triangle AOP are given all the angles and the side AP, whence OP and AO (HO) are known: then in the triangle OHK will be given the sides OH and OK (OP + PK), whence HK will be given; next, in the triangle BQH will be given QH and BQ (BP — HK), whence QBH, and its double QFH, will be given; lastly, in the triangle EFH are given all the angles and the perpendicular QH, whence the sides will also be given.

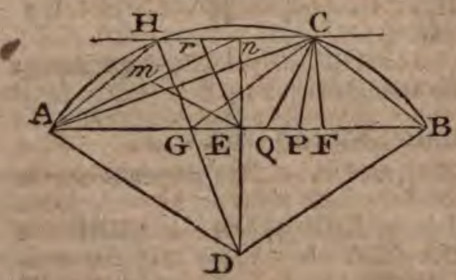
But the answer may be more easily brought out, by first finding HOK, the difference of the angles ABH and BAH, as in the fifth problem.

PROBLEM LXI.

The sum of the three sides, the difference of the angles at the base, and the length of the line bisecting the vertical angle of any plane triangle being given; to describe the triangle.

CONSTRUCTION.

Make AB equal to the sum of the sides, which bisect in E by the perpendicular DEn, and make the angle nEr equal to half the given difference of the angles at the base, taking Er equal to the line bisecting the vertical angle: through r draw Cnr parallel to AB, cutting DEn in n; draw nA, to which draw Em = Er, and draw AD parallel to Em, meeting nED in D; and on the center D, at the distance



of DA, describe the circle ACB, cutting rnC in C; join A, C, and B, C, and make the angle $BCF = CBF$; also make $ACG = CAG$, and let CF and CG meet AB in F and G; then will FCG be the triangle that was to be described.

DEMONSTRATION.

Upon AB let fall the perpendicular CP; let CQ bisect the vertical angle GCF, and let DH be drawn parallel to Er, meeting Cr in H. Then, by reason of the parallel lines, it will be as $Er : DH \ (:: En : Dn) :: Em : DA$; whence, Er being $= Em$ (*by construction*), DH and DA are also equal, and the point H falls in the periphery of the circle: therefore the angle nDH (nEr) at the center, standing upon half the arch HC, will be equal to the angle HAC, at the periphery, standing upon that whole arch, that is, equal to the difference of the angles ABC and BAC; but the angle GFC being double to ABC, and FGC double to BAC (*by construction*), the difference of GFC and FGC will be double to the difference between ABC and BAC, and therefore equal to $2nEr$ $(2nDH)$, the difference given. Moreover, because $GCQ = FCQ$, 2PCQ will be the difference between PCG and PCF, which must likewise be equal to $2nEr$, the difference of their complements PGC and PFC; whence $PCQ = nEr$, and consequently $CQ = Er$. Furthermore, since the angle $ACG = CAG$, and $BCF = CBF$, thence will $CG = AG$, and $CF = FB$; and therefore $CG + GF + FC = AB$. *Q. E. D.*

Method of Calculation.

In the triangle Enr are given all the angles and the side Er, whence En will be given; next, in the triangle AEn will be given (besides the right angle) both the legs En and EA, whence the angle EnA is given; then it will be, as the radius to the sine of DHn or Ern $(:: DH : Dn :: DA : Dn)$ so is the sine of DnA to the sine of DAn; whence ADn, the supplement of ACB, is also given, from which all the rest of the angles in the figure are given by addition and subtraction only.

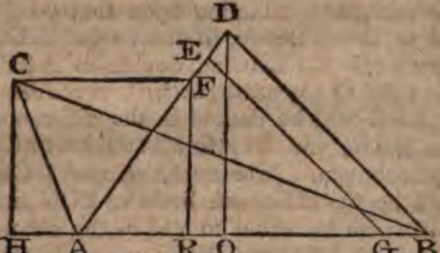
This method of solving the problem, it may be observed, requires three operations by the sines and tangents; but the same thing may be performed by two proportions only: for as Er : AE : : the secant of rEn : the tangent of EnA; whence all the rest will be found as above.

PROBLEM LXII.

To reduce a given triangle into the form of another, or to make a triangle which shall be similar to one triangle, and equal to another.

CONSTRUCTION.

Upon the base AB of the triangle ABC, to which you would make another triangle equal, describe ADB similar to the triangle required; draw CF parallel to AB, meeting AD in F; take AE a mean proportional between AD and AF; and, parallel to DB, draw EG; then will AGE be the triangle that was to be constructed.



DEMONSTRATION.

Let FR and DQ be perpendicular to AB; then the triang. ADB : triang. ACB : : DQ : FR (*schol. Euc.* 1. 6.) : : AD : AF (*Euc.* 4. 6.) : : AD^2 : AD × AF (*Euc.* 1. 6.) : : AD^2 : AE^2 (*by construction*) : : triang. ADB : triang. AEG (*Euc.* 19. 6.). Therefore the antecedents of the first and last of these equal ratios being the same, the consequents ACB and AEG must necessarily be equal. *Q. E. D.*

Method of Calculation.

In the triangle ADB are given all the angles and the side AB, whence AD will be given; next, in the triangle AFR will be given all the angles and the side

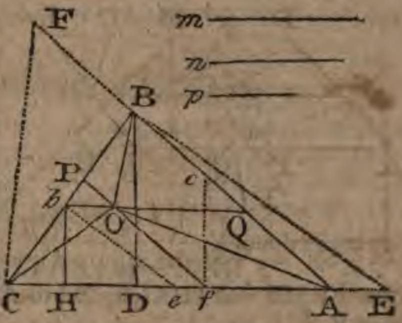
FR (= CH), whence AF will be given; and then AD and AF being given, $AE = \sqrt{AD \times AF}$ will also be given.

PROBLEM LXIII.

To find a point in a given triangle ABC, *from whence right lines drawn to the three angular points, shall divide the whole triangle into parts* (COA, AOB, BOC) *having the same ratio one to another, as three given right lines m, n, and p, respectively.*

CONSTRUCTION.

In CA and AB produced, if need be, take CE and AF, each equal to $m + n + p$, joining E, B, and F, C; take $Ce = m$, $Ac = n$, and draw eb and cf parallel to EB and CF, meeting the sides of the given triangle in b and f; draw also bQ and fP parallel to AC and AB, and at O, the intersection of these lines, will be the point required.



DEMONSTRATION.

Let bH and BD be perpendicular to AC. The triangles CBE, Cbe, as also CBD, CbH, are similar; therefore, m (Ce) : $m + n + p$ (CE) : : Cb : CB : : bH : BD : : the triangle AOC : triangle ABC. In the very same manner it may be proved, that the part AOB is to the whole triangle ABC, as n to $m + n + p$; whence it follows, that the remaining part BOC must be to the whole triangle, as p to $m + n + p$; therefore these parts are to one another in the given ratio of m, n, and p. *Q. E. D.*

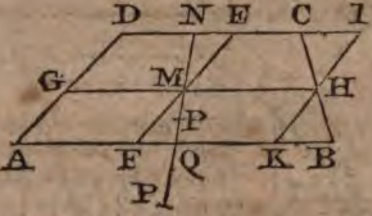
Method of Calculation.

Since $\begin{cases} m+n+p : m :: AB : AQ, \\ m+n+p : n :: AC : Af\,(QO), \end{cases}$
both AQ and QO will be given from thence; then, in the triangle AOQ, will be given two sides, and the included angle, from which every thing else will be known.

PROBLEM LXIV.

To divide a given trapezium ABCD, *whose opposite sides* AB, CD *are parallel, according to a given ratio, by a right line* QN, *passing through a given point* P, *and falling upon the two parallel sides.*

CONSTRUCTION.



Bisect AD in G, and draw GH parallel to AB (or DC), meeting BC in H: then divide GH in M, according to the given ratio, and through M draw PQN, and the thing is done.

DEMONSTRATION.

Draw EMF and IHK parallel to AD, meeting DC and AB in E, I, K, and F.

Because of the parallel lines, we have GD = ME = HI, and AG = FM = KH; whence, as GD is = AG (*by construction*), ME will be = FM, and HI = HK; and the triangle EMN will be = FMQ, and IHC = BHK (*Euc.* 4. 1.); whence it appears that the trapezium AQND is also equal to the parallelogram DF, and the trapezium QBCN equal to the parallelogram FI; but these parallelograms are to one another as their bases, or as GM to MH (*Euc.* 1. 6.); therefore GM : MH : : ANQD : QBCN. *Q. E. D.*

Method of Calculation.

Whereas AB and DC are parallel, GH is an arithmetical mean between them, and therefore equal to half their sum. Therefore, as the whole line GH and the ratio of its parts GM, MH are given, the parts themselves will also be given.

PROBLEM LXV.

To cut off from a given trapezium ABCD, *whose opposite sides* AB, CD *are parallel, a part* AQND *equal to a rectangle given, by a right line passing through a given point* P, *and falling upon the two parallel sides.* (*See the figure to the last problem.*

CONSTRUCTION.

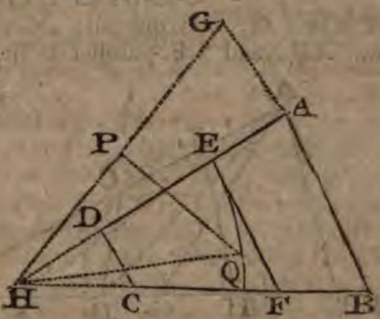
Bisect AD in G, and draw GH parallel to AB; upon AD (*by Euc.* 45. 1.) describe the parallelogram ADEF equal to the rectangle given, and through the intersection of GH and EF draw PQN, and the thing is done. The demonstration whereof is manifest from the preceding problem.

PROBLEM LXVI.

To divide a given trapezium ABCD, *whose sides* AB *and* DC *are parallel, into two equal parts, by a right line parallel to those sides.*

CONSTRUCTION.

Produce AD and BC till they meet in H, and make AG equal and perpendicular to HD; draw HG, and bisect the same with the perpendicular PQ = HP; join H, Q, and in HA take HE, equal to HQ, and parallel to AB draw EF, and the thing is done.



DEMONSTRATION.

Since HE^2 $\left(= HQ^2 = HP^2 + PQ^2 = 2HP^2 = \frac{HG^2}{2} = \frac{HA^2 + AG^2}{2} = \frac{HA^2 + HD^2}{2}\right)$ is an arithmetical mean between HA^2 and HD^2, it is evident that the triangle HEF will also be an arithmetical mean between the triangles HAB and HDC (or ABFE = EFCD); because those triangles, being similar, are to one another as (HE^2, HA^2, HD^2) the squares of their homologous sides. *Q. E. D.*

Method of Calculation.

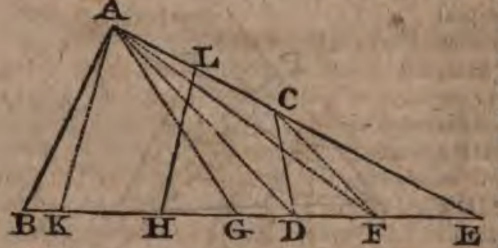
Since all the sides and angles of the trapezium are supposed given, the side CD and all the angles of the triangle HDC will be given; therefore HD and AH will be known; whence HE, $= \sqrt{\dfrac{HD^2 + HA^2}{2}}$, will also be given. But the same thing may be had without the angles; for, since DC is parallel to AB, we have AB — DC : AD : : DC : HD; whence HE will be given, as before.

PROBLEM LXVII.

To divide a given trapezium ABCD *according to a given ratio, by a right line* LH *cutting the opposite sides* AC, BD *in given angles.*

CONSTRUCTION.

Produce the said opposite sides till they meet in E; draw AD, and CF parallel to it, meeting BE in F; divide BF in G, according to the given ratio; and, having made EAK equal to the given angle which LH is to make with AC, take EH a mean proportional between EG and EK; then draw HL parallel to AK, and the thing is done.



DEMONSTRATION.

By construction, EG : EH : : EH : EK : : EL : EA (*Euc.* 5. 6.); whence it follows that EG × EA =

EH × EL, and consequently that the triangles EHL and EAG are also equal to each other (*Euc.* 15. 6.), from which taking away EDC, common, the remainders CDHL and CDGA will be likewise equal, and consequently ALHB = AGB, being the differences between those remainders and ACDB. But the triangle ADF is = ACD, standing upon the same base AD and between the same parallels; therefore (by adding AGD, common) AGF is also = CDGA (= CDHL); but AGF (CDHL) : AGB (ALHB) : : GF : GB (*Euc.* 1. 6.). *Q. E. D.*

Method of Calculation.

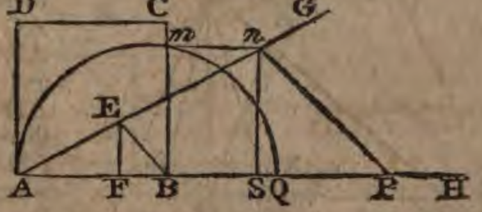
In the triangles ABE and ABK are given all the angles and the side AB, whence BE, BK, and EC will be known; then in the triangle EFC will be given all the angles and the side CE, whence EF, and from thence FG, and EG, will be known; lastly, from the known values of EK, EG, and EF, the value of FH ($= \sqrt{EG \times EK} - EF$) will be found.

PROBLEM LXVIII.

Two right lines AG *and* AH, *meeting in a point* A, *being given by position; it is required to draw a right line* *n*P *to cut those lines in given angles, so that the triangle* A*n*P, *formed from thence, may be equal to a given square* ABCD.

CONSTRUCTION.

Let the angle ABE be equal to the given angle AP*n*, and let BE meet AG in E; draw EF perpendicular to AH, make BQ equal 2EF, and upon AQ describe the semicircle A*m*Q, cutting BC in *m*; draw *mn* parallel to AH, meeting AG in *n*, and *n*P parallel to EB, and A*n*P will be the triangle required.



DEMONSTRATION.

The triangles AEB and A*n*P, being similar, are to one another as the squares of their perpendicular heights EF and *m*B (*n*S): but mB^2 is $= BQ \times AB = 2EF \times AB$; therefore it will be, as the triangle AEB ($EF \times \frac{1}{2}AB$) : the triangle A*n*P : : $EF^2 : 2EF \times AB$: : $EF : 2AB$: : $EF \times \frac{1}{2}AB : AB^2$ (*Euc.* 1. 6.); wherefore, the antecedents being the same, the consequents must necessarily be equal, that is, $AnP = ABCD$. *Q. E. D.*

Method of Calculation.

In the triangle ABE are given all the angles and the side AB, whence EF will be given, and consequently S*n* ($= \sqrt{AB \times 2EF}$); whence AP and A*n* are also given.

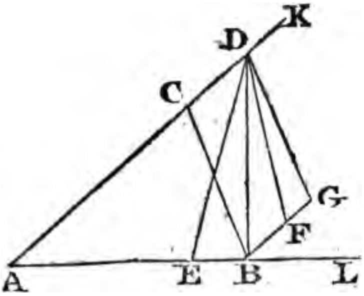
LEMMA.

If from any point C, *in one side of a plane angle* KAL, *a right line* CB *be drawn, cutting both sides* AK, AL *in equal angles* (ACB, ABC); *and from any other point* D *in the same side* AK *another right line be drawn, to cut off an area* ADE *equal to the area* ABC; *I say, that* DE *will be greater than* CB.

DEMONSTRATION.

Complete the parallelogram DCBG, and join B, D, and in BG (produced if need be) take BF = BE, and draw FD.

Since the triangles ABC, AED are equal, by supposition, and have one angle, A, common, therefore will AD : AC : : AB (AC) : AE (*Euc.* 15. 6.), and consequently AD + AE greater than AC + AB (*Euc.* 25. 5.); whence it is manifest that CD must be greater than EB, or BG than BF. Moreover, because the angle ABC



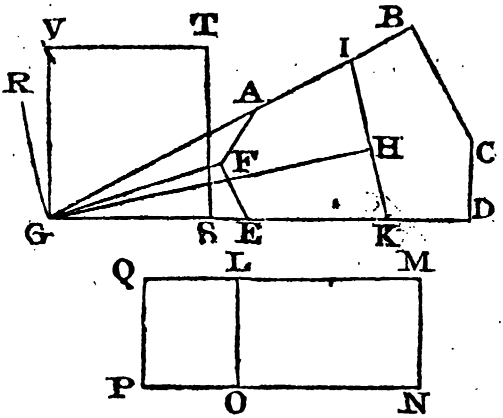
(= ACB = CDG) is = GBC, it will be greater than GBD, which is but a part of GBC; and therefore ABD must, evidently, be greater than GBD; wherefore, seeing BF and BE are equal, and that DB is common to both the triangles DBE, DBF, it is manifest that DE is greater than DF (*Euc.* 19. 1.); but DF is greater than DG (*by the same*), because the angle DGF (DCB), being obtuse, is greater than GFD, which must be acute (*Euc.* 32. 1.): consequently DE is greater than DG, or its equal CB. *Q. E. D.*

PROBLEM LXIX.

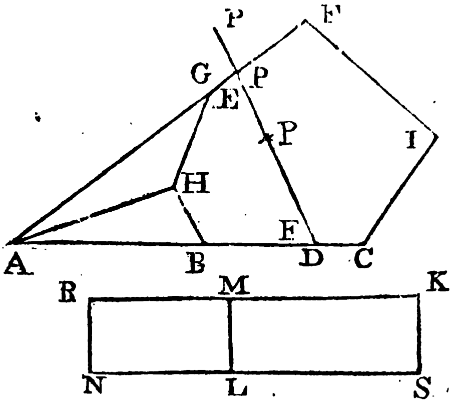
From a given polygon ABCDEF, *to cut off a given area* AFEIK, *by the shortest right line*, KI, *possible.*

CONSTRUCTION.

Let the given area to be cut off be represented by the rectangle LMNO; and let the sides AB and DE, by which the dividing line is terminated, be produced till they meet in G; make upon OL (*by Euc.* 45. 1.) a rectangle OQ equal to AFEG, and let a square GSTV be constituted (*by Euc.* 14. 2.) equal to the whole rectangle QN: bisect the angle BGD by the right line GH, and make GR perpendicular to GH; and draw



MN equal to AGHB, and, by the last problem, let ED be so drawn through the given point P, that the triangle AED, formed from thence, may be equal to the whole



rectangle KN; then will EDBHG be equal to KL: for since AED is = KN, let the equal quantities AGHB and MN be taken away, and there will remain EDBHG = KL.

Method of Calculation.

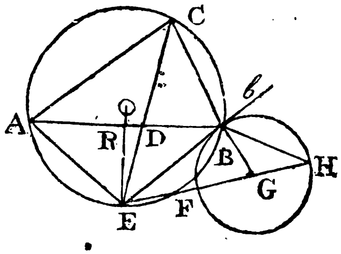
Let the area of the figure AGHB be found, by dividing it into triangles, and let this area be added to the area given, and the sum will be equal to the area AED, or the rectangle KN; from whence AD will be found, as in the last problem.

PROBLEM LXXII.

Having the base, the vertical angle, and the length of the line bisecting that angle and terminating in the base, to describe the triangle.

CONSTRUCTION.

Upon the given base AB let a segment of a circle ACB be described (*by problem* 4) to contain the given angle, and, having completed the whole circle, from O, the center thereof, perpendicular to AB, let the radius OE be drawn; draw EB, and make BG perpendicular thereto, and equal to half the given bisecting line, and from G, as a center, with the radius GB, let a circle BHF be described, intersecting EG (when drawn) in F and H; from E to AB draw ED = EF, and let the same be produced to meet the circumference in C; join A, C, and B, C; so shall ABC be the triangle required.



DEMONSTRATION.

The triangles CBE and BDE are similar, because the angle BEC is common to both, and the angles BCE and DBE stand upon equal arches BE and AE; therefore EC : EB : : EB : ED, and consequently ED × EC = EB^2: but (*by Euc.* 36. 3.) EB^2 = EF × EH = ED × EH (*by construction*). Hence ED × EC = ED × EH, and consequently EC = EH; from which taking away the equal quantities ED and EF, there remains DC = FH = the given line bisecting the vertical angle (*by construction*): and it is evident that DC bisects the angle ACB, since ACD and BCD stand upon equal arches AE and EB. *Q. E. D.*

Method of Calculation.

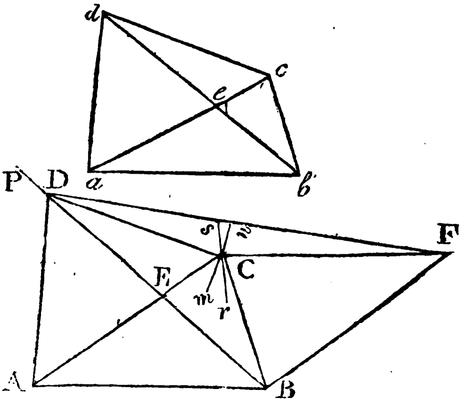
If BE be considered as a radius, BR ($\frac{1}{2}$AB) will be the co-sine of the angle EBR, and BG the tangent of BEG; therefore BR : BG (or AB : DC) : : co-sine EBR (ACE) : tang. BEG, whose half complement EHB is likewise given from hence: then the angle HB*b* (supposing EB produced to *b*) being the complement of EHB, we shall have tang. EHB : rad. (: : sine EHB : co-sine EHB : : BE : EH : : EB : EC) : : sine ECB : sine CBE = sine EDB = co-sine OED, half the difference of the angles (ABC and BAC) at the base.

PROBLEM LXXIII.

Having given the two opposite sides ab, cd, the two diagonals ac, bd, and also the angle aeb in which they intersect each other; to describe the trapezium.

CONSTRUCTION.

In the indefinite line BP take BD equal to *bd*, and make the angle DBF equal to the given angle *aeb*, and BF = *ac*; also from the centers D and F, with the radii *dc* and *ab*,



let two arches *m*C*n* and *r*C*s* be described intersecting each other in C; join D, C, and F, C, and make BA equal and parallel to FC; then draw AD, AC, and BC, and the thing is done.

DEMONSTRATION.

Since (*by construction*) AB is equal and parallel to CF, therefore will AC be equal and parallel to BF (*Euc.* 33. 1.), and consequently the angle AEB (*Euc.* 29. 1.) = DBF = *aeb*. *Q. E. D.*

Method of Calculation.

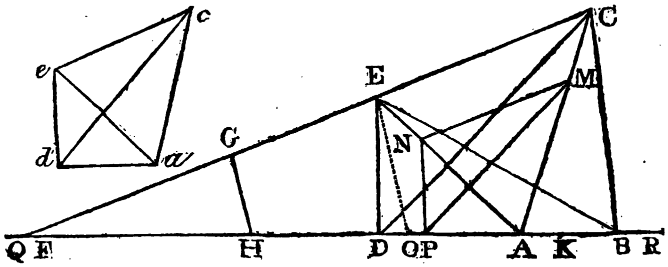
Join D, F; then in the triangle DBF will be given two sides DB, BF, and the angle included; whence the angle BFD and the side DF will be known; then in the triangle DFC will be given all the three sides; whence the angle DFC will be known, from which BFC (BFD — DFC) = BAC will also be known.

PROBLEM LXXIV.

Having given the two diagonals and all the angles, to describe the trapezium.

CONSTRUCTION.

Assume AB at pleasure; and, having produced the same both ways, make the angles QAC, RBC equal, respectively, to two opposite angles *a* and *e* of the trapezium; moreover, make ACF equal to *ace*, one of the remaining angles; and from F, the intersection of AF and BQ, take FG = the given diagonal *dc*, and draw GH parallel to CB, meeting FB in H. Then from A and B (*by the lemma, p.* 336) let two lines AE and BE be drawn to meet in FC, so as to be in the given ratio of *ae* to FH; in AE take AN = *ae*, and draw NM parallel to FC, meeting AC in M; lastly, draw NP making the angle MNP = *ced*, and meeting FB in P; so shall AMNP be the true figure required.



DEMONSTRATION.

Let ED be parallel to NP, and let DC and PM be drawn.

It is evident, by construction, that the diagonal AN, and all the angles of the trapezium, are equal to the respective given ones; it therefore remains only to prove that PM is equal to the other given diagonal *dc*. Now the angle RBC being = CED (*by construction*), the circumference of a circle may be described through all the four angular points of the trapezium BCED; and so the triangles FBE and FCD (as both the angles FBE and FCD stand upon the same chord ED) will be similar; and consequently BE : DC (: : FB : FC) : : FH : FG (*dc*). But (*by construction*) AE : BE : : *ae* : FH; therefore, by compounding these two proportions, we have AE : DC : : *ae* : *dc*; but (because of the similar figures ADEC, APNM) we also have AE : DC : : AN (*ae*) : PM; and consequently PM = *dc*. *Q. E. D.*

Method of Calculation.

All the angles of the triangles ABC, FAC, and FBC being given, we shall have sine ACB × sine F : sine ABC × sine ACF : : AB : AF; and sine FHG (FBC) : sine FGH (FCB) : : FG (*dc*) : FH; whence AF and FH are known.

Find $AK = \frac{AB \times ae}{FH + ae}$, and $KO = \frac{AK \times FH}{FH - ae}$: which last is equal to (OE) the radius of the circle determining the point E (*see the aforesaid lemma*). Therefore, in the triangle FOE are given two sides FO and OE, besides the angle F; whence the angle FOE will be given; then in the triangle AOE will be given OA, OE, and the included angle; whence the angle OAE, which the diagonal AN makes with the side AP, will be known, and from thence every thing else required.

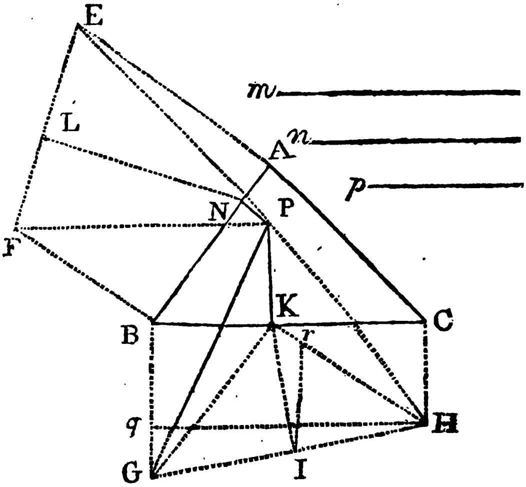
This problem, as the circle described from O cuts FC in two points, admits of two different solutions (except, only, when FC touches the circle). If the circle neither cuts nor touches that line, the problem will be impossible; the limits of the ratio of AE to BE (and consequently of *ae* to *dc*) growing narrower and narrower, as AB becomes less and less, with respect to AC, or according as the sum of the opposite angles ($a + e$ = QAC + RBC) approaches nearer and nearer to two right angles; so that, at last (supposing AC and BC to coincide), AE and BE will be, every where, in the ratio of equality; therefore *cd* can here have only one particular ratio to *ae*; and the diagonal ANE may be drawn at pleasure, the problem being, in this case, indeterminate.

PROBLEM LXXV.

Supposing the right lines m, n, p to represent the lengths of three staves erected perpendicular to the horizon, in the given points A, B, C; *to find a point* P, *in the plane of the horizon* ABC, *equally remote from the top of each staff.*

CONSTRUCTION.

Join A, B, and B, C, and make AE and BF perpendicular to AB; also make BG and CH perpendicular to BC, and let AE be taken $= m$, CH $= p$, and BF and BG each $= n$; draw EF and GH, which bisect by the perpendiculars LN and IK, cutting AB and CB in N and K; make KP and NP perpendicular to BC and BA, and the intersection P of those perpendiculars will be the point required.



DEMONSTRATION.

Conceive the planes AEFB and BCHG to be turned up, so as to stand perpendicular to the plane of the horizon ABC and intersect it in the right lines AB and BC; then, because BF and BG are equal to each other, and perpendicular to the plane of the horizon, it is evident that the points F and G must coincide, and that AE, BG (BF), and CH will represent the true position of the staves: suppose KG, KH, PG, PH, PE, and PF to be now drawn; then, since (*by construction*) GI = HI, and the angle GIK = HIK; therefore is GK = HK (*Euc.* 4. 1.): moreover, since KP is (*by construction*) perpendicular to BC, it will also be perpendicular to the plane BCHG, and consequently the angles PKG and PKH both right angles: therefore, seeing the two triangles GKP, HKP have two sides and an included angle equal, the remaining sides PG and PH must likewise be equal (*Euc.* 4. 1.). After the very same manner it is proved that PF (or PG) is equal to EP. *Q. E. D.*

Method of Calculation.

Draw Ir perpendicular, and Hq parallel to BC; then, by reason of the similar triangles HqG and IrK, it will be as BC (Hq) : BG — CH (Gq) : : $\frac{BG + CH}{2}$ (Ir) : Kr = $\frac{\overline{BG - CH} \times \overline{BG + CH}}{2BC}$; which subtracted from Br (= $\frac{1}{2}$BC) gives BK: and in the same manner will BN be found; then in the trapezium KBNP will be given all the angles and the two sides BK and BN; from whence the remaining sides, &c. may be easily determined.

PROBLEM LXXVI.

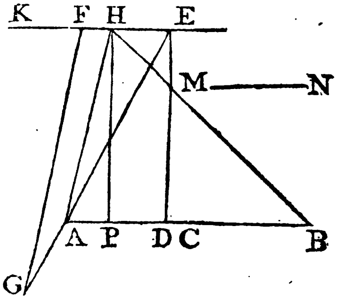
The base, the perpendicular, and the difference of the sides being given, to determine the triangle.

CONSTRUCTION.

Bisect the base AB in C, and in it take CD a third proportional to 2AB and the given difference of the sides MN; erect DE equal to the given perpendicular, and draw EK parallel to AB, and take therein EF = MN; draw EAG, to which, from F, apply FG = AB; draw AH parallel to FG, meeting EK in H: then draw BH, and the thing is done.

DEMONSTRATION.

By reason of the parallel lines, FG (AB) : FE (MN) : : AH : EH (DP); therefore AB × DP = AH × MN, or 2AB × DP = 2AH × MN; to which last equal quantities adding 2AB × CD = MN^2, (*by construction*) we have 2AB × CP = 2AH × MN × MN^2; but 2AB × CP is = $\text{BH}^2 - \text{AH}^2$ (*by a known property of triangles*); therefore $\text{BH}^2 - \text{AH}^2 = 2\text{AH} \times \text{MN} + \text{MN}^2$, or $\text{BH}^2 = \text{AH}^2 + 2\text{AH} \times \text{MN} + \text{MN}^2 = \overline{\text{AH} + \text{MN}}|^2$ (*Euc.* 4. 2.), consequently BH = AH + MN. *Q. E. D.*



Method of Calculation.

In the right-angled triangle ADE we have DE and AD $\left(= \frac{1}{2}\text{AB} - \frac{\text{MN}^2}{2\text{AB}}\right)$; whence the angle DAE (FEG) will be found; then in the triangle EFG will be given two sides and one angle, from which the angle GFK (= BAH) will also be known.

PROBLEM LXXVII.

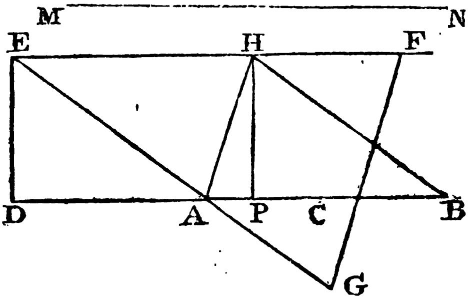
The base, the perpendicular, and the sum of the two sides being given, to describe the triangle.

CONSTRUCTION.

Bisect the base AB in C, and in it produced take CD a third proportional to 2AB, and the sum of the sides, MN; erect DE equal to the given perpendicular, and draw HE parallel to AB, and take therein EF = MN; draw EAG, to which, from F, apply FG = AB; draw AH parallel to FG, meeting EF in H; then draw BH, and the thing is done.

DEMONSTRATION.

Because of the parallel lines, FG (AB) : FE (MN) : : AH : EH (DP); and therefore $2\mathrm{MN} \times \mathrm{AH} = 2\mathrm{AB} \times \mathrm{DP}$; which equal quantities being subtracted from $\mathrm{MN}^2 = 2\mathrm{AB} \times \mathrm{CD}$, (*by construction*) there will remain $\mathrm{MN}^2 - 2\mathrm{MN} \times \mathrm{AH} = 2\mathrm{AB} \times \mathrm{CP} = \mathrm{BH}^2 - \mathrm{AH}^2$; whence, by adding AH^2 to each, we have $\mathrm{MN}^2 - 2\mathrm{MN} \times \mathrm{AH} + \mathrm{AH}^2 = \mathrm{BH}^2$, that is, $\overline{\mathrm{MN} - \mathrm{AH}}|^2 = \mathrm{BH}^2$; therefore $\mathrm{MN} - \mathrm{AH} = \mathrm{BH}$, or $\mathrm{MN} = \mathrm{BH} + \mathrm{AH}$. *Q. E. D.*



Method of Calculation.

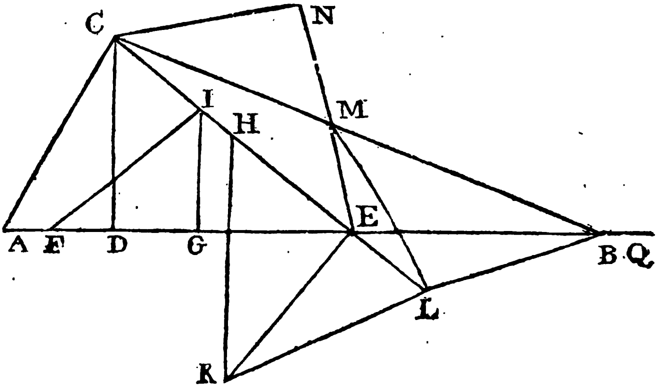
In the triangle AED are given (besides the right angle) both the legs; whence the angle DAE (= FEG) will be given; then in the triangle FEG one angle and two sides will be known, from which the angle EFG (= BAH) will be determined.

PROBLEM LXXVIII.

The difference of the two sides, the perpendicular, and the vertical angle being given, to determine the triangle.

CONSTRUCTION.

Upon the indefinite line FEQ erect the given perpendicular DC, making the angle DCE = half the given angle; let EF, expressing the given difference of the sides, be bisected by the perpendicular GI, meeting EC in I; also let EC be bisected in H, and make EK perpendicular to CE, and equal to EI; and, having drawn HK, take HL, in HE produced, equal thereto; from L to FQ apply LB = EK, and join C, B; also



draw EM, making the angle CEM equal to DEC, and cutting CB in M; then from C to QEF apply CA = CM, so shall ACB be the triangle required.

DEMONSTRATION.

Upon EM let fall the perpendicular CN, and join L, M, and F, I. Now, $LB^2 = EK^2$ (*by construction*) $= \overline{HK + HE} \times \overline{HK - HE}$ (*Euc.* 5. 2.) $= \overline{HL + CH} \times \overline{HL - HE}$ (*by construction*) $= CL \times EL$; whence CL : LB : : LB : EL; therefore the triangles CLB and ELB must be equiangular (*Euc.* 6. 6.), and consequently LBM = LEB = CED = CEM (*by construction*). Therefore, since the external angle CEM of the trapezium LEMB, is equal to the opposite internal angle B, the circumference of a circle will pass through all the four angular points; and consequently the angle LMB will be = LEB, both standing upon the same chord LB; but it is proved that LBM is = LEB; therefore LMB = LBM = FEI; and so the triangles BLM and EIF, being isosceles, and having LMB = EFI, and also LB = EI (*by construction*), they will be equal in all respects, and consequently BM = EF; whence BC — AC (= BC — CM = BM) = EF, the given difference (*by construction*). Moreover, CEN being = CED (*by construction*), CN will be = CD; and so CM being = CA, ACD will be = MCN, to which adding DCM, common, we have ACB = DCN = 2DCE. *Q. E. D.*

Method of Calculation.

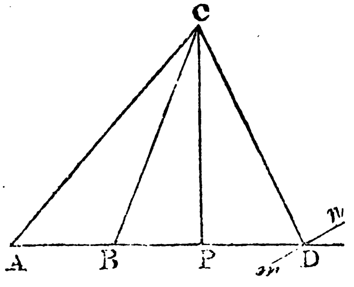
Seeing EG and EH are the sine and tangent of EIG and EKH, to the equal radii EI and EK, it will therefore be EG : EH (or EF : EC) : : sine EIG (ECD) : tangent EKH. But EC : CD : : the radius : co-sine ECD; whence, by compounding these proportions, EF : CD : : radius × sine ECD : co-sine ECD × tangent EKH : : $\frac{\text{radius} \times \text{sine ECD}}{\text{co-sine ECD}}$ (= tangent ECD) : tangent EKH; from which EKL, half the complement of EKH will be also given; then it will be, as the radius : tangent EKL (: : KE : EL : : LB : EL) : : sine LEB (CED) : sine LBE (BCE); which proportions, expressed in words, give the following theorem:

As the difference of the sides is to the perpendicular, so is the tangent of half the vertical angle to the tangent of an angle; and as the radius is to the tangent of half the complement of this angle, so is the co-sine of half the vertical angle to the sine of half the difference of the angles at the base.

PROBLEM LXXIX.

The perpendicular, the difference of the sides, and the difference of the angles at the base being given, to determine the triangle.

CONSTRUCTION.



Let a triangle ABC be constructed *by the last problem*, whose perpendicular and difference of the sides shall be the same with those given, and whereof the vertical angle ACB is also equal to the given difference of angles;

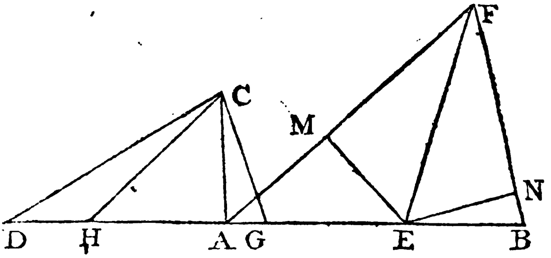
then upon C, as a center, with the radius CB, let an arch be described, intersecting AB, produced, in D; join C, D, and ACD will be the triangle required. For CD being = CB, the angle CDB will also be = CBD = A + BCA (*Euc.* 32. 1.). The method of calculation is also the same as in the preceding problem.

PROBLEM LXXX.

The perpendicular, the sum of the two sides, and the vertical angle being given, to describe the triangle.

CONSTRUCTION.

Upon AB, the given sum of the two sides, erect AC equal to the given perpendicular; and make the angle ACH equal to the complement of half the given angle; upon AB (*by prob.* 72) let a triangle ABF be



constituted, whose vertical angle, AFB, shall be equal to the given one, and whereof the bisecting line FE (terminating in the base) shall be = DC; then draw CG and CH parallel to FB and FA, so shall GCH be the triangle required.

DEMONSTRATION.

It is evident that the angle HCG is = AFB = the given one. Moreover, if EM and EN be taken as perpendiculars to AF and BF, they will be equal to each other, and also equal to the given one AC, because all the angles EFN, EFM, and ADC are equal, by construction, and EF is likewise = CD; whence, as the angles AHC, AGC are respectively equal to EAM, EBN, it is evident that HC = EA, and GC = EB, and consequently that HC + GC (= EA + EB) = AB. *Q. E. D.*

Method of Calculation.

By the problem above referred to, AB : CD (EF) : : co-sine ADC (AFE) : tangent of an angle; which let be denoted by Q.

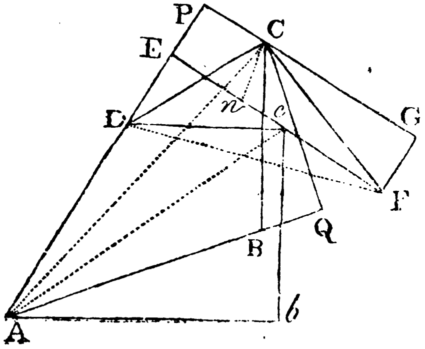
Now, CD : CA : : radius : sine ADC; which proportion being compounded with the former, we have AB : CA : : co-sine ADC × radius : tangent Q × sine ADC : : $\frac{\text{co-sine ADC} \times \text{radius}}{\text{sine ADC}}$ (co-tangent ADC) : tangent Q. Then, by the same problem, it will be as tangent $\frac{1}{2}$Q : radius : : sine ADC : co-sine of the difference of the angles (G and H) at the base. The above proportions, given in words at length, exhibit the following theorem:

As the sum of the sides is to the perpendicular, so is the co-tangent of half the vertical angle to the tangent of an angle; and, as the tangent of half this angle is to the radius, so is the sine of half the vertical angle to the co-sine of half the difference of the angles at the base.

PROBLEM LXXXI.

To constitute a trapezium of a given magnitude under four given lines.

CONSTRUCTION.



Make a right angle *b* with two of the given lines A*b*, *bc*; and with the other two complete the trapezium A*bc*D: upon AD let fall the perpendicular *c*E, in which produced (if necessary) take EF, so that the rectangle under it, and AD, may be double the given area: moreover, take a fourth proportional to AD, A*b*, and *bc*, with which, from the center F, let an arch be described, meeting another arch, described from D with the radius D*c* in C; join D, C; and from A and C draw the other two given lines AB, CB, so as to meet, and they will thereby form the trapezium ABCD, as required.

DEMONSTRATION.

Draw A*c*, AC, and FC; upon AD and AB let fall the perpendiculars CP, CQ; and make FG perpendicular to PCG.

Because $AD^2 + DC^2 + 2AD \times DP$ ($= AC^2$, *Euc.* 12. 2.) $= AB^2 + BC^2 + 2AB \times BQ$, and $AD^2 + Dc^2 + 2AD \times DE$ ($= Ac^2$) $= Ab^2 + bc^2$ (*Euc.* 47. 1.), it follows, by taking these last equal quantities from the former, that $2AD \times DP - 2AD \times DE$ $(2AD + EP) = 2AB \times BQ$, and consequently that BQ : EP (FG) : : AD : AB : : BC : FC (*by construction*); whence the triangles BCQ, FCG are similar, and so CQ : CG : : BC : FC : : AD : AB (*by construction*), and therefore $CQ \times AB = CG \times AD$; hence, by adding $CP \times AD$ to each, we have $CP \times AD + CQ \times AB$ (= twice the area ABCD) $= CP \times AD + CG \times AD = EF \times AD$ = twice the given area (*by construction*). *Q. E. D.*

Method of Calculation.

From DE $\left(= \frac{Ab^2 + bc^2 - AD^2 - Dc^2}{2AD}\right)$ and EF $\left(= \frac{2\text{ area}}{AD}\right)$ the value of DF, and likewise that of the angle ADF, will be found; then, all the sides of the triangle DCF being known, the angle FDC will likewise be known; which, added to ADF, gives (ADC) one of the angles of the trapezium.

It may so happen that a trapezium, having one right angle, cannot be constituted under the four given lines; in which case it will be necessary (instead of forming the trapezium A*bc*D) to lay down AD first, and in it (produced if needful) to take DE equal to

$\frac{\overline{AB}|^2 + \overline{BC}|^2 - \overline{AD}|^2 - \overline{DC}|^2}{2AD}$, that is, equal to the altitude of a rectangle, formed on the base 2AD, whereof the contained area is equal to the difference of $\overline{AB}|^2 + \overline{BC}|^2$ and $\overline{AD}|^2 + \overline{DC}|^2$ (which line DE is to be set off on the other side of D, when the latter of these two quantities is the greater): this being done, the rest of the solution will remain the same, as is manifest from the first and second steps of the demonstration; the process from thence to the end being nowise different.

It may be further observed, that the problem itself becomes impossible, when the two circles, described from the centers D and F, neither cut nor touch; the greatest limit of the area, and consequently of EF, being when they touch each other; in which case the sum of the radii DC, FC becoming = DF, the point C will fall in the line DF, and the angle DCF will become equal to two right angles; but the sum of the opposite external angles CDP and CBQ is always equal to DCF; because CDP (supposing C*n* parallel to AP) is = DC*n*, and CBQ (= CFG) = FC*n*: hence it is evident that the limit, or the greatest area will be when the sum of the opposite angles is equal to two right angles, or when the trapezium may be inscribed in a circle.

FINIS.